Misellaneous Exercises

1(a) Show that if A is a connected subset of a metric space X, then the closure $cl(A)$ of A is also connected

(b) Show by example that $cl(A)$ can be connected even though A is not connected.

2. If X is a metric space and $x \in X$, define C_x = the union of all the connected subsets of X that contain x. Prove:

(a) C_x is connected

(b) C_x is closed

(c) If $C_x \cap C_y \neq \phi$, then $C_x = C_y$.

Conclude: (with terminology C_x is a "component" of X):

(α) X is a union of nonempty closed connected set, each of which is maximal relative to connectedness.

(β) If there are only finitely many components, there the components are open

(γ) In general, components need not be open (give an example).

3. If U is an open subset of $\mathbb{R}$, then U is a disjoint union of countably many (nonempty) open intervals. Prove this!

4. Let C = the Cantor set $\subset [0,1]$ (Open middle thirds removed $[0,1] - (\frac{1}{3},\frac{2}{3}) - (\frac{1}{9},\frac{2}{9}) - (\frac{7}{9},\frac{8}{9}) \ldots$).

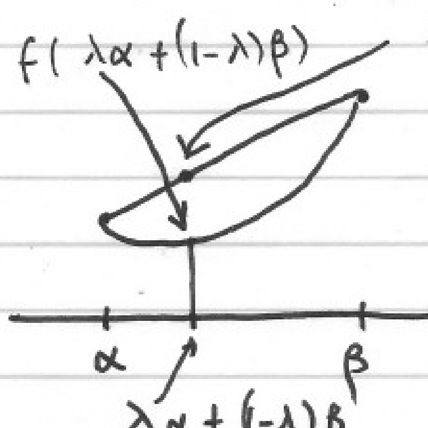
(a) What is the sum of the lengths of the finite-length components of $\mathbb{R} - C$?

(b) Construct, given $\varepsilon \in (0,1)$, a closed subset C_ε of $[0,1]$ that has sum of the lengths of the finite-length components of $\mathbb{R} - C_\varepsilon = \varepsilon$ and the "structure" of C_ε is like C (that is C_ε is obtained by removing open intervals situated in the centers of each remaining closed interval).

(c) Show that all closed subsets of $[0,1]$ with this structure are uncountable.
(You may use Baire Category, but it is instructive to try to prove this directly: note that it is enough to do C itself!).

5. (a) Prove that $\sin x < x$ for all $x > 0$.
(b) Prove that $\tan x > x$ for all $x \ni 0 < x < \pi/2$
(c) Deduce that for $x \in (0, \pi/2)$, $x \cos x < \sin x < x$
(d) Illustrate this graphically (in $\tan x \geq x$ and $\sin x < x$ terms).

6. (a) Show that if $f'' > 0$ on (a,b) then for $\alpha, \beta \in (a,b)$ and $\lambda \in (0,1)$
$$f(\lambda\alpha + (1-\lambda)\beta) < \lambda f(a) + (1-\lambda) f(b)$$



(b) Prove $f'' \geq 0 \Rightarrow$
$$f \leq \lambda f + (1-\lambda) f$$
[Suggestion: Apply (a) to $f + \varepsilon x^2$, $\varepsilon > 0$ and take a limit with α, β, λ fixed as $\varepsilon \to 0^+$]

For 6(a), you will need: problem 7:

7. If f is differentiable on (a,b), if $f(x_0) \geq f(x)$ for all $x \in (a,b)$ and if $f''(x_0)$ exists (i.e. f' is differentiable at x_0) then $f''(x_0) \leq 0$. (Suggestion: If $f''(x_0) > 0$, prove $f' > 0$ on $(x_0, x_0+\varepsilon)$ for ε small enough).

8(a) Prove that $\sum_{n=1}^{+\infty} \frac{1}{n!} \sin(nx)$ is C^∞ on $\mathbb{R}$.

(b) Prove that $\sum_{n=1}^{+\infty} \frac{1}{n^2} \sin(n!x)$

is continuous on $\mathbb{R}$. Is it differentiable at $x=0$?

9. Prove that if $f:(0,1) \to \mathbb{R}$ is three-times differentiable with $f^{(3)}(x) \equiv 0$ for all $x \in (0,1)$, then f is a polynomial of degree ≤ 2.

10. Prove carefully that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \cdots$ by integrating $\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} \ldots$

11 Prove that: if X is a sequentially compact metric space then X is (covering) compact by filling in the following outline: With X seq. compact:

(a) X has a countable dense subset (done for Arzela Ascoli Th.)

(b) If $S = \{s_1, \ldots s_n \ldots\}$ and $\mathcal{B}$ = the set of balls $B(s_j, r)$ $r \in \mathbb{Q}$ $r > 0$, $j = 1, \ldots 2$ has the property that every open set U in X = union of balls in $\mathcal{B}$ that are $\subseteq U$.

(c) Every open cover of X has a countable subcover (Same as $\mathbb{R}^n$ case proof)

(d) If $\bigcup_{n=1}^{+\infty} U_n = X$ then $\exists N \ni \bigcup_{1}^{N} U_n = X$ (proof: If not, choose $x_N \notin \bigcup_{1}^{N} U_n$, $x_{N_j} \to x_0$. But $x_0 \in U_n$ some n etc.