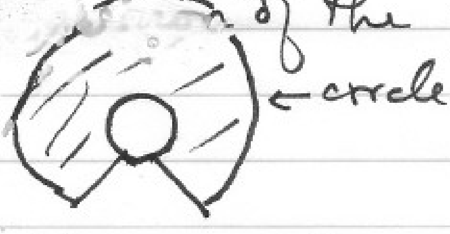
Assignment III

Multivariable Calculus Exercises

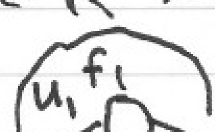
1. An open set $U \subset \mathbb{R}^2$ is <u>simply connected</u> if each closed differential form on U is exact (i.e. for each $P, Q : U \to \mathbb{R}$, C^1 functions with $\partial P/\partial y = \partial Q/\partial x$, then is a differentiable function $f : U \to \mathbb{R}$ with $\partial f/\partial x = P$ and $\partial f/\partial y = Q$). Prove that if U_1 and U_2 are simply connected and $U_1 \cap U_2$ is connected, then $U_1 \cup U_2$ is simply connected.

2. An open set $U \subset \mathbb{R}^2$ is <u>star-shaped</u> if $\exists$ $p_0 \in U$ such that for all $p \in U$ the line segment $\{\lambda p_0 + (1-\lambda)p : \lambda \in [0,1]\} \subset U$. Show that if U is star-shaped, then U is simply connected. (Suggestion: Define f by integrating $P\,dx + Q\,dy$ along the line segment from p_0 to p to get $f(p)$. Show that $\partial f/\partial x = P$ and $\partial f/\partial y = Q$ by noting that the line segment $\subset$ open rectangle $\subset U$).

3. Combine problems 1 and 2 to prove that an open set of the form shown is simply connected.

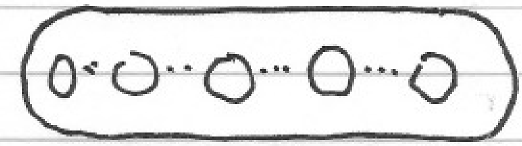


4. Using problem 3, show that $\frac{\text{closed}}{\text{exact}}$ for $U = \{(x,y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$ is one-dimensional. (Suggestion



Wolog $f_1 = f_2$ on right-hand piece of $U_1 \cap U_2$. $\exists f$ on $U \Leftrightarrow f_1 = f_2$ on left piece.)

5. Generalize problem 4 to show that



k holes, $k \geq 1$, has $\dfrac{\text{closed}}{\text{exact}} \cong \mathbb{R}^k$.

6. Find explicit (P, Q) generators (a basis, actually) for the $\cong \mathbb{R}^k$ vector space of problem 5
(Suggestion: Recall "$d\theta$ but no θ" on $\mathbb{R}^2 - \{\vec{0}\}$).

7. A C^1 function $f : U^{\text{open}} \to \mathbb{R}^2$, $U \subset \mathbb{R}^2$, is <u>holomorphic</u> if $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$ (where $f(x,y) = (u(x,y), v(x,y))$).
Use the binomial theorem to show that $f(z) = z^n$, $z \in \mathbb{C} \triangleq \mathbb{R}^n$ is holomorphic.

8. Use the theorems we have proved about power series (in $z \in \mathbb{C}$) to show that if $\sum_{n=0}^{+\infty} a_n z^n$ has radius of convergence $R > 0$, then $\sum a_n z^n$ is holomorphic on $\{z : |z| < R\}$.

9. Prove that if $f : U^{\text{open}} \to \mathbb{R}$ is C^2 and holomorphic, then u and v are harmonic, i.e. $\Delta u \equiv 0$ on U, and $\Delta v \equiv 0$ on U where $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

10. Show that if u is C^2 and harmonic ($\mathbb{R}$ valued) on a simply connected open set $U \subset \mathbb{R}^2$, then $\exists$ a (C^1) function $v : U \to \mathbb{R}$ such that
$f = (u, v)$ is holomorphic on U.

11. Show by example that simple connectivity can be required in problem 10, i.e., exhibit a U and a harmonic $u : U \to \mathbb{R}$ for which no such v exists (Suggestion: $\log r$)

12. Let $U \subset \mathbb{R}^3$ be an open rectangle and $W: U \to \mathbb{R}^3$ a C^2 vector field on U with $\operatorname{div} W = 0$. Show that $\exists$ a (C^1) vector field $V: U \to \mathbb{R}^3$ with $\operatorname{curl} V = W$ (Suggestion: Look, with $W = (A, B, C)$ for V in the form $(v_1(x,y,z), v_2(x,y,z), 0)$. So need $-\frac{\partial v_2}{\partial z} = A(x,y,z), \frac{\partial v_1}{\partial z} = B(x,y,z), \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = C(x,y,z)$

Find $v_1(x,y,0)$ and $v_2(x,y,0)$ to solve last equation. Then get $v_1(x,y,z)$ and $v_2(x,y,z)$ to get v_1, v_2. Use $\operatorname{div} W = 0$ condition to check $\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} = C$ for all x, y, z by checking that $\frac{\partial}{\partial z}(\text{LHS}) = \frac{\partial C}{\partial z}$)

13. Assume the 2-dimensional divergence theorem" in the form (for discs)

$$\int_\gamma \langle V, N \rangle ds = \int_{\text{interior of } \gamma} \operatorname{div} V \, d(\text{area})$$

where the left hand side is the arclength integral over a circle γ, N = exterior unit normal of γ and the area integral is over the interior of the circle, V being a vector field C^1 on $U \supset \gamma \cup (\text{interior of } \gamma)$. Use this to prove the "Mean Value Theorem for Harmonic Functions": If f is harmonic on $U \subset \mathbb{R}^2$, $\vec{0} \in U$ and $\{(x,y) : x^2+y^2 \leq R^2\} \subset U$, then

$$\frac{1}{2\pi} \int_0^{2\pi} f(r\cos\theta, r\sin\theta)\, d\theta$$

is independent of $r \in (0, R]$ and hence $= f(0)$.