Problem Set I

1. Suppose $\{x_n\}$ is a bounded sequence (i.e., $\exists M>0 \ni |x_n| \le M$ for all n). Show that $\{x_n\}$ has a convergent subsequence by showing that there is a subsequence that converges to a greatest lower bound of $\{\lambda \in \mathbb{R} : x_n < \lambda$ for all but a finite number of n-values$\}$.

2. Recall (Tao, p 117) that a real number is by definition an equivalence class of Cauchy sequences of rational numbers. Suppose A is a nonempty set of real numbers which has an upper bound, i.e., $\exists\, \alpha$ such that $x \le \alpha \;\; \forall x \in A$. Define b_k = the smallest rat. no. of the form p/b_k, $p \in \mathbb{Z}$, such that $x \le b_k$, $\forall x \in A$. $(k = 1, 2, 3 \ldots)$. Show that

$$b_1 \ge b_2 \ge b_3 \ldots \qquad \text{and that } \{b_k : k = 1, 2, 3 \ldots\}$$

is a Cauchy sequence. Then show that the real no. β that is the equivalence class of this C. sequence is the least upper bound of A.

3. Let a and b be positive real numbers, $a > b$. Set $a_0 = a$, $b_0 = b$, and for $j = 1, 2, \ldots$

$$a_j = \tfrac{1}{2}(a_{j-1} + b_{j-1}) \qquad b_j = \sqrt{a_{j-1} b_{j-1}}\,.$$

Show that

(1) $a_{j+1} \le a_j$, $b_{j+1} \ge b_j$ $\quad \forall j = 0, 1, 2, 3 \ldots$

(2) $a_j \ge b_j$, $\quad \forall j = 0, 1, 2, 3 \ldots$

(3) Every $a_j \ge b_k$, $j, k \in \{0, 1, 2, 3 .\}$

(4) (Use problem 4 to deduce that) $\lim\limits_{j\to+\infty} a_j$ and $\lim\limits_{j\to+\infty} b_j$ both exist and $\lim\limits_j a_j = \lim b_j$.

4. Suppos $a_1 \le a_2 \le a_3 \le a_4 \ldots$, $a_j \in \mathbb{R}$. Prove that if $\exists M \ni a_j \le M$, $\forall j$, then $\lim a_j$ exists and equals lub $\{a_j : j = 1, 2, 3 \ldots\}$.

5(a) Prove that for each positive real number a, there are nonnegative integers $N_0, N_1, N_2, N_3 \ldots$ with $0 \le N_j \le 9$, $j = 1, 2, 3 \ldots$ such that

$$a = \text{lub}\left\{ N_0 + \sum_{j=1}^{k} N_j 10^{-j} : k = 1, 2, 3 \ldots \right\}$$

(b) Show that the N's are unique subject to the conditions given except for the cases "like" $.099999\ldots = .1$

6. Show that a real number a is rational if and only if its "decimal expansion" (from problem 5) is eventually periodic in the sense that from some j onward the sequence $N_0\ N_1\ N_2 \ldots$ consists of a single finite block repeated infinitely.

7(a) Let $\{a_j\}$ be a sequence that is bounded above in the sense that $\exists M \ni a_j \le M$, $\forall j$. Use problem 4 to show that, if $A_n \overset{\text{def.}}{=}$ least upper bound of $\{a_j : j \ge n\}$, then $\lim A_n$ exists.

(b) Prove that $\lim A_n$ is the largest number which is the limit of some convergent subsequence of $\{a_j\}$.

Notation $\lim A_n$ is denoted $\limsup \{a_j\}$.

8. Prove that if $\{a_j\}$ is a sequence in $\mathbb{R}$, then the set L consisting of all limits of subsequences of $\{a_j\}$ is closed. Use this fact to give an alternative definition of $\limsup \{a_j\}$ (cf., the second statement in problem 7).

9. Define a sequence $\{a_j\}$ of real numbers as follows

$$\pi = 3.141592\ldots$$

$$a_1 = \pi - 3$$
$$a_2 = 10(\pi - 3.1)$$
$$a_3 = 100(\pi - 3.14)$$
$$\vdots$$

so that $a_1 = .141592\ldots$, $a_2 = .41592\ldots$, $a_3 = .1592\ldots$, $a_4 = .592\ldots$, etc.

Show that $\{a_j\}$ has a convergent subsequence.

10. Set $a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n+1)$, $n = 1, 2, 3 \ldots$

(a) Show that $a_n \le a_{n+1}$ for all $n = 1, 2, 3 \ldots$

[Suggestion: One gets from a_n to a_{n+1} by adding $\frac{1}{n+1} + \ln(n+1) - \ln(n+2)$ (Use that $\ln(n+2) - \ln(n+1) = \int_{n+1}^{n+2} \frac{1}{t}\,dt$ to show this is positive!]

(b) If $b_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln n$ then $b_{n+1} \le b_n$, $n = 1, 2 \ldots$

(c) Deduce that $\{a_n\}$ is bounded above so $\lim a_n$ exists.

(from $a_n < b_n$ ↘)