Sample Basic Examination Analysis Solutions
Spring (March) 2008

1. (i) If $g(a) = a$ and/or $g(b) = b$, we are done. So assume $g(a) > a$ and $g(b) < b$. [Note that $g(a) \geq a$ and $g(b) \leq b$ by hypothesis]. Set
$$F(x) = g(x) - x \quad , x \in [a,b].$$
Then $F(a) > 0$ but $F(b) < 0$. The function F is continuous since g is, so, by the Intermediate Value Theorem for continuous functions, $\exists\ x_0 \in (a,b)$ such that $F(x_0) = 0$. Then $g(x_0) - x_0 = 0$ or $g(x_0) = x_0$. And x_0 is thus the desired fix point.

(ii) This is the Contraction Mapping Theorem in a special case. Uniqueness is clear since $g(x) = x$ and $g(y) = y$ implies that
$$|g(x) - g(y)| = |x-y|.$$
This is inconsistent with $|g(x) - g(y)| \leq \gamma |x-y|$, $\gamma < 1$, unless $|x-y| = 0$, or $x = y$.

For existence, choose x_0 arbitrarily (as indicated) and define x_n inductively by $x_{n+1} = g(x_n)$. (as the problem states). Then
$$\left\{ \begin{aligned} |x_{n+2} - x_{n+1}| &= |g(x_{n+1}) - x_{n+1}| = |g(x_{n+1}) - g(x_n)| \\ &\leq \gamma\ |x_{n+1} - x_n| \quad \text{by hypothesis.} \end{aligned} \right\}$$
Put $C = |g(x_0) - x_0|$. Then induction gives $|x_{n+1} - x_n| \leq C\gamma^n$: this is true by choice of C when $n=0$, and $\{\ \}$ calculation is the inductive step.

Hence, if $n_1 > n_2$, $|x_{n_1} - x_{n_2}|$

$$\leq |x_{n_1} - x_{n_1-1}| + \dots + |x_{n_2+1} - x_{n_2}|$$

$$\leq C(\gamma^{n_1-1} + \gamma^{n_1-2} + \dots + \gamma^{n_2}) \leq C(\gamma^{n_2} + \gamma^{n_2+1} + \dots)$$

$= C\gamma^{n_2}/(1-\gamma)$. This $\to 0$ as $n_2 \to \infty$. So $\{x_n\}$ is a Cauchy sequence. Its limit, call it x_∞ is in $[a,b]$ since $[a,b]$ is closed. Also $g(x_\infty) = \lim_{n\to\infty} g(x_n)$ since g is continuous. But $g(x_n) = x_{n+1}$ so $\lim_{n\to+\infty} g(x_n) = \lim_{n\to+\infty} x_{n+1} = \lim x_n = x_\infty$

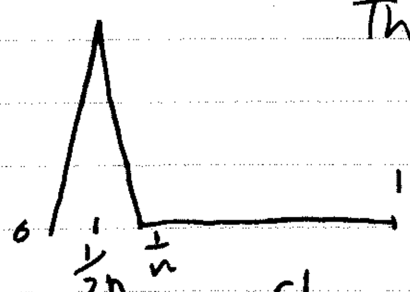
and x_0 is a fixed point, as required.

2. The statement is false. Let $f_n(x)$, $n = 1,2,3,4\ldots$ be defined by

$$f_n(x) = 0 \quad \text{if } x > \tfrac{1}{n}$$

$$f_n(x) = 2nx \quad \text{if } x \in [0, \tfrac{1}{2n})$$

$$f_n(x) = 1 - 2n(x - \tfrac{1}{2n}) \quad \text{if } x \in [\tfrac{1}{2n}, \tfrac{1}{n}].$$



Then f_n is continuous since left hand limit = right hand limit = value at $\frac{1}{2n}$ and at $\frac{1}{n}$ (the only questionable points).

$$\int_0^1 f_n(x)\,dx = \frac{1}{2}\left(\frac{1}{n}\right)(2n) = 1.$$ area of triangle!

But $\lim f_n(0) = 0$ since $f_n(0) = 0$, all n while for $x > 0$, $f_n(x) = 0$ if $n > \frac{1}{x}$. So $\lim f_n(x) = 0$. all x.

Thus $\int_0^1 \lim f_n = \int_0^1 0 = 0$

while $\lim \int_0^1 f_n = 1$.

3. Since it is given that f is C^4, it is appropriate to expand $f(x+h)$ and $f(x-h)$ in Taylor form (around x) with expansion up to third derivative and remainder, namely

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} f^{(4)}(\lambda)$$

where $\lambda \in (x, x+h)$. (This is Taylor's formula with remainder). Similarly,

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f^{(3)}(x) + \frac{h^4}{4!} f^{(4)}(\mu)$$

where $\mu \in (x-h, x)$.

Thus $\dfrac{f(x+h) + f(x-h) - 2f(x)}{h^2} =$ (calculate directly) $f''(x) + \dfrac{h^4}{4!}\left(f^{(iv)}(\lambda) + f^{(iv)}(\mu)\right)$

So the |error| of the approximation is

$$= h^{-2}\left|\frac{h^4}{4!}\left(f^{(iv)}(\lambda) + f^{(iv)}(\mu)\right)\right|$$

$$\leq \frac{2h^2}{4!} \sup_{\lambda \in [x-h, x+h]} |f^{(iv)}(\lambda)|$$

When h is small, this is $\dfrac{h^2}{12}|f^{(iv)}(x)| + h^2 o(h)$

by the continuity of $f^{(iv)}$.

($f(x) = x^4$: example to show constant needed, etc.)
(optimal)

4. Given $\varepsilon > 0$, and $x \in X$, choose n_x (depends on x) such that $f_{n_x}(x) < \varepsilon/2$. Then choose an open set U_x with $x \in U_x$ such that $y \in U_x \Rightarrow |f_{n_x}(x) - f_{n_x}(y)| < \varepsilon/2$. This is possible by the continuity of f_{n_x}. Note $|f_{n_x}(y)| < \varepsilon$ then.

Consider the open cover of X, $X = \bigcup_{x \in X} U_x$. By compactness, there is a finite subcover, $X = U_{x_1} \cup U_{x_2} \cup \ldots \cup U_{x_k}$. Let $N_\varepsilon = \max(n_{x_1}, \ldots, n_{x_k})$, the n's as already chosen. Now suppose $N \geq N_\varepsilon$. We claim that then $f_N(y) < \varepsilon$ (and of course $0 \leq f_N(y)$) for all $y \in X$. If this claim is true, then N_ε suffices for the given $\varepsilon > 0$ in the definition of uniform convergence to 0.

To check the claim, note that $y \in U_{x_j}$ for some $j = 1, 2 \ldots, k$. So, looking at the fifth line of our solution ("Note $|f_n(y)| < \varepsilon$..") we have that

$$|f_{n_{x_j}}(y)| < \varepsilon \quad \text{so} \quad 0 \leq f_{n_j}(y) < \varepsilon.$$

But $N \geq N_\varepsilon = \max(n_{x_1}, \ldots, n_{x_k}) \geq n_j$. So monotonicity as given implies

$$0 \leq f_N(y) \leq f_{n_j}(y) < \varepsilon. \qquad \square$$

5(a) If $F(x_0,y_0) > 0$, by continuity there is a closed square S (with positive side length) centered at (x_0,y_0) on which $F \geq \frac{1}{2}F(x_0,y_0) > 0$ (choose $\varepsilon = \frac{1}{2}F(x_0,y_0)$ in definition of continuity). Then

$$\iint_S F \geq (\text{area of } S)\left(\tfrac{1}{2}F(x_0,y_0)\right) > 0,$$

contradicting the hypotheses. A similar argument show $F(x_0,y_0) < 0$ is impossible (on a square centered at (x_0,y_0), $F < \frac{1}{2}F(x_0,y_0) < 0$ etc.)

(b) For any square S, (x_0,y_0), (x_0+h,y_0), (x_0,y_0+h) (x_0+h,y_0+h) vertices,

$$\int_{x_0}^{x_0+h}\left(\int_{y_0}^{y_0+h}\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)dy\right)dx = \int_{x_0}^{x_0+h}\left(\frac{\partial f}{\partial x}\Big|_{(x,y_0+h)} - \frac{\partial f}{\partial x}\Big|_{(x,y_0)}\right)dx$$

$$= \overbrace{f(x_0+h,y_0+h) - f(x_0,y_0+h)}^{\text{integral of first term}}$$

$$- \overbrace{\left(f(x_0+h,y_0) - f(x_0,y_0)\right)}^{\text{from second term}}$$

$$= f(x_0+h,y_0+h) + f(x_0,y_0) - f(x_0,y_0+h) - f(x_0+h,y_0)$$

Calculation gives similarly that

$$\int_{y_0}^{y_0+h}\left(\int_{x_0}^{x_0+h}\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)dx\right)dy = $$ this same four-term expression.

Thus, using that iterated integrals = double integral:

$$\iint_S \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \iint_S \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \quad \text{or} \quad \iint_S\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\right] = 0.$$

Part (a) $\Rightarrow$ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$ everywhere on $\mathbb{R}^2$ $\square$

6. If X is countble, say $X = \{x_1, x_2 \ldots\}$ then

$$X = \bigcup_{j=1}^{+\infty} \{x_j\} \quad \left(\text{union of "singletons"}\right)$$

By the Baire Category Theorem, if X is complete, then one of the sets, "singletons" $\{x_j\}$, $j = 1, 2, ..$ must fail to be nowhere dense. But a singleton is always closed in a metric space. So for $\{x_j\}$, some j, to fail to be nowhere dense, it must be that it has nonempty interior (since it is closed). So the set set $\{x_j\}$ must be an open set. (The point x_j must be isolated). □

7. For each $n = 1, 2, 3, ..$ let $S_n = \{x : a(x) \geq \frac{1}{n}\}$. The set S_n contains no more than Mn elements, since if $x_1, \ldots, x_k$ were distinct elements of S with $k > Mn$, then $a(x_1) + \ldots + a(x_k)$ would be $\geq k(1/n) > M$. [Here no more than Mn elements means no more than "greatest integer $\leq Mn$ elements"]. So S_n is finite. But

$$\{x : a(x) > 0\} = \bigcup_{n=1}^{+\infty} S_n$$

since for each positive α, there is an n with $\frac{1}{n} \leq \alpha$. So $\{x : a(x) > 0\}$ is a countable union of finite sets and is hence countable.