CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof

Bridging Matrix Recovery Gaps using Manifolds

Deanna Needell

Claremont McKenna College Joint work with Y. C. Eldar [Technion], Y. Plan [Univ. of Michigan]

ANTC Seminar, Claremont, Jan. 2012

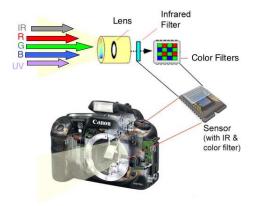
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Outline					

- Compressed Sensing (CS)
 - Applications
 - Mathematical Formulation
 - Best known results
- CS's sister: Matrix recovery
 - Applications
 - Mathematical Formulation
 - Best known results
- Comparison of the two problems
 - The question unanswered
 - Our answer
 - Proof via manifold theory

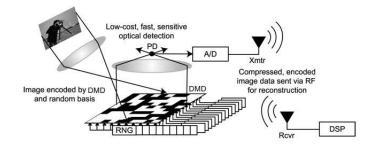
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Applications					
Digital Ca	meras				

Today's digital cameras already "old school?"



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Applications					
Digital Ca	meras				



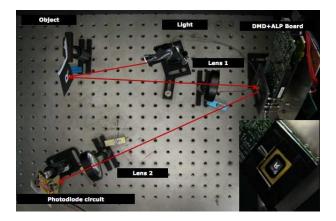
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Applications					

Digital Cameras

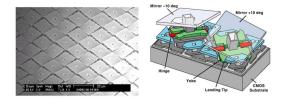
Save your nickels to buy the new digital camera?



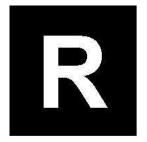
Deanna Needell Bridging Matrix Recovery Gaps using Manifolds

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Applications					
Digital Ca	meras				



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(Original)



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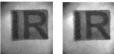
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Applications					
Digital Ca	meras				





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Applications					
MRI					

Feeling claustrophobic?

It'll only last a quick 45 minutes...



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Applications					
MRI					

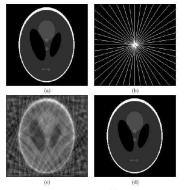


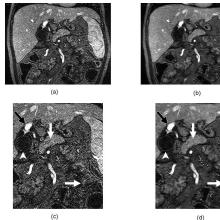
Figure 1: Example of a simple recovery problem. (a) The Logan-Shepp phanotn test image. (b) Sampling domain Ω in the frequency phase, Fourier coefficients are sampled long 22 approximately radial lines. (c) Minimum energy reconstruction obtained by satting unobserved Fourier coefficients to zero. (d) Reconstruction obtained by minimizing the total variation, sei in (1.1). The reconstruction is an easter treplica of the image in (a).

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Applications					

Pediatric MRI



(a-d) Submillimeter near-isotropic-resolution contrast-enhanced T1-weighted MR images in 8-year-old boy. (a, c) Standard and (b, d) compressed sensing reconstruction images. (c, d) Zoomed images show improved delineation of the pancreatic duct (vertical arrow), bowel (horizontal arrow), and gallbladder wall (arrowhead), and equivalent definition of portal vein (black arrow) with L1 SPIR-IT reconstruction.

(Caffey Award : Faster Pediatric MRI Via Compressed Sensing - Shreyas Vasanawala et.al. (Stanford University)) ≣⊳ Ξ DQC

CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Applications					
Many mor	e				

- Radar
- Error Correction
- Computational Biology (DNA Microarrays)
- Geophysical Data Analysis
- Data Mining, classification
- Neuroscience
- ...

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Mathematical Formula	ation				
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1 Signal of interest $f \in \mathbb{R}^d$

- 2 Measurement matrix $A : \mathbb{R}^d \to \mathbb{R}^m$.
- 3 Measurements y = Af.



Problem: Reconstruct signal f from measurements y

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Mathematical Formul	ation				

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Mathematical Formula	ation				
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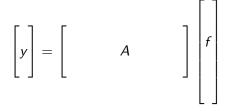


④ Problem: Reconstruct signal f from measurements y

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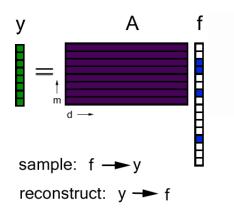
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4 Problem: Reconstruct signal f from measurements y

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Mathematical Formula	ation				

Without further assumptions, this problem is ill-posed.

Why will this work?

Most signals of interest contain far less information than their dimension d suggests.

Assume *f* is sparse:

• In the coordinate basis: $||f||_0 \stackrel{\text{def}}{=} |\operatorname{supp}(f)| \leq s \ll d$. In practice, we encounter compressible signals, and the measurements have noise. (Not in this talk.)

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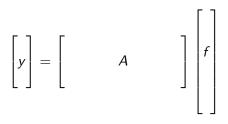
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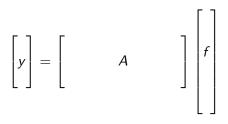


Important Questions

- What kind(s) of measurement matrices A?
- How many measurements needed?
- Are the guarantees uniform?
- Is algorithm stable?
- Fast runtime?

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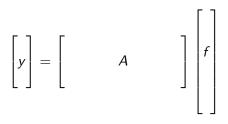


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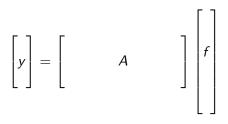


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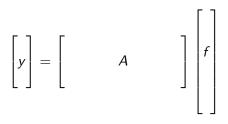
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Mathematical Formul	ation				
ℓ_0 -optimiz	ation				

The First CS Theorem

Let *A* be one-to-one on *s*-sparse vectors and set:

$$\hat{f} = \operatorname*{argmin}_{g} \|g\|_{0}$$
 such that $Ag = y$.

Then in the noiseless case, we have perfect recovery of all *s*-sparse signals: $\hat{f} = f$.

Proof: Easy!

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Moral of the story:

Theoretically, we need only m = 2s measurements.

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Moral of the story: *Theoretically*, we need only m = 2s measurements.

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Mathematical Formul	ation				
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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formulation	on				

Just relax: ℓ_1 -optimization

Relaxation [Candès-Tao]

Let A satisfy the *Restricted Isometry Property* for 2*s*-sparse vectors and set:

$$\hat{f} = \underset{g}{\operatorname{argmin}} \|g\|_{1}$$
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Proof:

(Not so easy)

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formulati	on				

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formulation	1				

 A satisfies the Restricted Isometry Property (RIP) when there is δ < c such that

 $(1-\delta)\|f\|_2\leq \|Af\|_2\leq (1+\delta)\|f\|_2$ whenever $\|f\|_0\leq s.$

 Gaussian or Bernoulli measurement matrices satisfy the RIP with high probability when

 $m \gtrsim s \log d$.

 Random Fourier and others with fast multiply have similar property: m ≥ s log⁴ d.

Moral of the story:

Practically, we need only $m = s \log d$ measurements.

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formulation	n				

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formul	ation				
The gap					

Problem:	CS
Theoretical	$\min \ f\ _0$
Practical	$\min \ f\ _1$
<i>m</i> for Practical	$m\gtrsim s\log n$
<i>m</i> for Theoretical	$m \ge 2s$

CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formul	ation				
The gap					

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CS Applications	CS Math 0000000●	MR Applications	MR Math	MR Theory	Proof
Mathematical Formul	ation				
The gap					

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CS Applications	CS Math 0000000●	MR Applications	MR Math	MR Theory	Proof
Mathematical Formul	ation				
The gap					

Problem:	CS	
Theoretical	$\min \ f\ _0$	
Practical	$\min \ f\ _1$	
<i>m</i> for Practical	$m\gtrsim s\log n$	
<i>m</i> for Theoretical	$m \ge 2s$	

CS Applications	CS Math 0000000●	MR Applications	MR Math	MR Theory	Proof
Mathematical Formul	ation				
The gap					

Problem:	CS
Theoretical	$\min \ f\ _0$
Practical	$\min \ f\ _1$
<i>m</i> for Practical	$m\gtrsim s\log n$
<i>m</i> for Theoretical	$m \ge 2s$

CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Applications					

The Netflix problem

That check is worth how much??



Deanna Needell Bridging Matrix Recovery Gaps using Manifolds

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Applications					

The Netflix problem

Tell us how you really feel...

Mo	vies You've F	late	d	
you've seen, ra you ma	on your 745 movie ratings, this is th seen. As you discover movies on th see them and they will show up on t y change the rating for any movie y move a movie from this list by click	Sort by > Star Rating + Jump to > 5 Stars +		
	TITLE	MPAA	GENRE	STAR RATING -
Add	12 Angry Men (1957)	UR	Classics	◎☆☆☆☆ @ Clear Rating)
Add	The 39 Steps (1935)	UR	Classics	◎☆☆☆☆ @ Clear Rating
Add	An American in Paris (1951)	UR	Classics	◎☆☆☆☆☆ @ Clear Rating
Add	The Andromeda Strain (1971)	G	Sci-Fi & Fantasy	◎ 숲 숲 숲 ☆ ☆ 한 Clear Rating
Add	Apollo 13 (1995)	PG	Drama	◎☆☆☆☆☆
Add	The Battle of Algiers (1965) La Battagla di Algeri	UR	Foreign	◎☆☆☆☆ @ Clear Rating
Add	Being There (1979)	PG	Drama	◎☆☆☆☆☆ @ Clear Rating
Add	Big Deal on Madonna Street (1958) I soliti ignoti	UR	Foreign	◎☆☆☆☆ @ Clear Rating
Add	The Birds (1963)	PG-13	Thrilors	◎☆☆☆☆☆
Add	Blade Runner (1982)	R	Sci-Fi & Fantasy	◎☆☆☆☆ Clear Rating

Deanna Needell

Bridging Matrix Recovery Gaps using Manifolds

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
		00000000			
Applications					

The Netflix problem

And we'll tell you how you really feel...

FOREIGN SUGGESTIONS (about 104) See all >



DRAMA SUGGESTIONS (about 82) See all >



Deanna Needell

Bridging Matrix Recovery Gaps using Manifolds

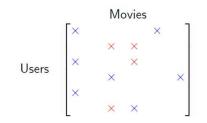
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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Applications					

Collaborative Filtering

We can use other people's preferences too, but still...



CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
Applications		00000000			
Surveilland	ce				

Separation of foreground and background!



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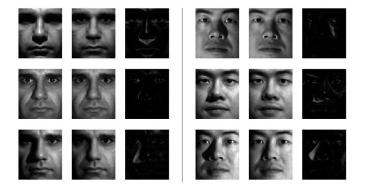
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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Applications					
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Computer Vision

Removing shadow and lighting effects!



Deanna Needell

Bridging Matrix Recovery Gaps using Manifolds DQC

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Applications					

Removing corruptions

And now enjoy the film ...



Repaired A



Corruptions

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
Applications					
Tilt [Cand	ès et.al.]				

For humans and computers who have trouble reading sideways...

Input (red window)



Output (rectified green window)



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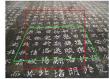
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Applications					
Tilt [Cand	ès et al l				

Fixing the leaning tower without any digging!

Т



Input (red window)



Output (rectified green window)









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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Mathematical Formula	tion				

The mathematical problem

1 Signal of interest $X \in \mathbb{R}^{n \times n}$

- 2 Linear measurement operator $\mathcal{A}: \mathbb{R}^{n imes n} o \mathbb{R}^m$.
- 3 Measurements $y = \mathcal{A}(X)$ of the form:

 $(\mathcal{A}(X))_i = \langle A_i, X \rangle = \operatorname{trace}(A_i^*X) \text{ for } A_i \in \mathbb{R}^{n \times n}$

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4 Problem: Reconstruct signal X from measurements y

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Mathematical Formula	ation				

Wait, isn't this impossible?

Without further assumptions, this problem is ill-posed.

Why will this work?

Most signals of interest contain far less information than their dimension $n \times n$ suggests.

Assume X is low-rank: rank $(X) \le r$. In practice, we encounter approximately low-rank signals, and the measurements have noise. (Not in this talk.)

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Mathematical Formula	tion				

Important Questions

- What kind(s) of linear operators \mathcal{A} ?
- How many measurements needed?
- Are the guarantees uniform?
- Is algorithm stable?
- Fast runtime?

Critical Connection

A matrix X is low-rank if and only if its vector $\sigma(X)$ of singular values is sparse!

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Rank opti	mization				

ℓ_0 -minimization

$$\hat{f} = \operatorname*{argmin}_{g} \|g\|_{0}$$
 such that $Ag = y$.

Rank-minimization

$$\hat{X} = \underset{M}{\operatorname{argmin}} \|\sigma(M)\|_0 = \underset{M}{\operatorname{argmin}} \operatorname{rank}(M) \text{ such that } \mathcal{A}(M) = y$$

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Relaxed optimization

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Nuclear norm minimization

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Nuclear norm

$$\|M\|_* = \|\sigma(M)\|_1 = \operatorname{trace}(\sqrt{M^*M})$$

Deanna Needell Bridging Matrix Recovery Gaps using Manifolds

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Deanna Needell Bridging Matrix Recovery Gaps using Manifolds

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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
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Nuclear norm optimization

Theorem [Oymak-Hassibi]

Let \mathcal{A} be a Gaussian linear operator and set,

$$\hat{X} = \underset{M}{\operatorname{argmin}} \|M\|_*$$
 such that $\mathcal{A}(M) = y$,

where $||M||_{*} = trace(\sqrt{M^{*}M}) = ||\sigma(M)||_{1}$.

Then in the noiseless case, to guarantee perfect recovery of any rank-r matrix X, we need only m = 16nr measurements.

Moral of the story:

Practically, we need only m = 16nr measurements.

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Problem:	CS	MR
Theoretical	$\ \min \ f\ _0$	min rank (X)
Practical	$\min \ f\ _1$	$\min \ X\ _*$
<i>m</i> for Practical	$m\gtrsim s\log n$	$m \ge 16nr$
<i>m</i> for Theoretical	$m \ge 2s$??

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The missing	ng gap				

The unanswered question

How many measurements m are needed to guarantee exact recovery of a rank-r matrix X via the rank minimization method?

$$\hat{X} = \operatorname{argmin}_{M} \operatorname{rank}(M)$$
 such that $\mathcal{A}(M) = y$

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Theoretical MR					

Answering the question

Theorem [Eldar-N-Plan]

Let $r \leq n/2$. When $\mathcal{A} : \mathbb{R}^{n \times n} \to \mathbb{R}^m$ is a Gaussian operator with $m \geq 4nr - 4r^2$, any rank-*r* (or less) matrix *X* is exactly recovered via rank minimization:

$$\hat{X} = \operatorname*{argmin}_{M} \operatorname{rank}(M)$$
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$\hat{X} = \operatorname{argmin}_{M} \operatorname{rank}(M)$ such that $\mathcal{A}(M) = y$

- Success of rank minimization is equivalent to asking that no rank-2*r* or less matrix resides in the kernel of *A*.
- Set $\mathcal{R} = \{X \in \mathbb{R}^{n \times n} : \operatorname{rank}(X) = 2r, \|X\|_F = 1\}.$
- \mathcal{R} is a smooth manifold of $4nr 4r^2 1$ dimensions.
- Step 1: Compute how large *m* must be to guarantee null(*A*) is disjoint from *R*.
- Step 2: Repeat for smaller values of the rank.

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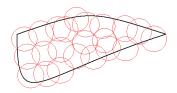
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CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof ○●○○○○
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Covering I	Numbers				

 For a set B, norm || · ||, and value ε, we define N(B, || · ||, ε) to be the smallest number of || · ||-balls of radius ε whose union contains B.



- The covering itself is called an ε-net.
- Euclidean covering numbers are well-known:

$$N(B_2^d, \|\cdot\|_2, \varepsilon) \le \left(\frac{3}{\varepsilon}\right)^d$$

Deanna Needell Bridging Matrix Recovery Gaps using Manifolds

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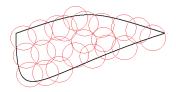
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- Fix *i*. ϕ^{-1} is Lipschitz: $\|\phi^{-1}(x) \phi^{-1}(y)\|_F \le L\|x y\|_2$.
- Let $\overline{B_2^d}$ be an (ε/L) -net for B_2^d of size at most $\left(\frac{3L}{\varepsilon}\right)^d$.
- Then $\overline{\mathcal{V}}$ defined by $\overline{\mathcal{V}} = \phi^{-1}(\overline{B_2^d})$ is an ε -net for \mathcal{V} .

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$$\mathcal{R} = \{X \in \mathbb{R}^{n \times n} : rank(X) = 2r, \|X\|_F = 1\}, d = 4nr - 4r^2 - 1$$

- Since \mathcal{R} is a smooth manifold, there is a countable partition $\{\mathcal{V}_i\}$ of closed sets with C^1 -diffeomorphisms $\phi_i : \mathcal{V}_i \to B_2^d$.
- Fix *i*. ϕ^{-1} is Lipschitz: $\|\phi^{-1}(x) \phi^{-1}(y)\|_F \le L\|x y\|_2$.
- Let $\overline{B_2^d}$ be an (ε/L) -net for B_2^d of size at most $(\frac{3L}{\varepsilon})^d$.
- Then $\overline{\mathcal{V}}$ defined by $\overline{\mathcal{V}} = \phi^{-1}(\overline{B_2^d})$ is an ε -net for \mathcal{V} .

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• Since $\overline{\mathcal{V}}$ is an ε -net,

$$\inf_{X\in\mathcal{V}}\|\mathcal{A}(X)\|_{\infty}\geq\min_{\overline{X}\in\overline{\mathcal{V}}}\|\mathcal{A}(\overline{X})\|_{\infty}-\varepsilon\cdot\|\mathcal{A}\|_{F\to\infty}.$$

• Therefore,

$$\mathbb{P}\Big(\inf_{X\in\mathcal{V}}\|\mathcal{A}(X)\|_{\infty}=0\Big)\leq \mathbb{P}\left(\inf_{X\in\mathcal{V}}\|\mathcal{A}(X)\|_{\infty}\leq \varepsilon\log(1/\varepsilon)\Big)\ \leq \mathbb{P}\left(\min_{\overline{X}\in\overline{\mathcal{V}}}\|\mathcal{A}(\overline{X})\|_{\infty}-\varepsilon\cdot\|\mathcal{A}\|_{F
ightarrow\infty}\leq \varepsilon\log(1/\varepsilon)\Big)\,.$$

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ight). \end{split}$$

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- $\leq \left(\frac{3L}{\varepsilon}\right)^d \cdot \prod_{i=1}^m \left(\mathbb{P}\left(|z_i| \leq 2\varepsilon \log(1/\varepsilon)\right)\right)$, (where z_i is an entry of $\mathcal{A}(\overline{X})$)
- $ullet \lesssim arepsilon^{m-d} \cdot (\log(1/arepsilon))^m$
- So we choose m = d + 1 and take $\varepsilon
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Problem:	CS	MR
Theoretical	$\min \ f\ _0$	min rank (X)
Practical	$\min \ f\ _1$	$\min \ X\ _*$
<i>m</i> for Practical	$m\gtrsim s\log n$	$m \ge 16 nr$
<i>m</i> for Theoretical	$m \ge 2s$	$4nr - 4r^2$

CS Applications	CS Math	MR Applications	MR Math	MR Theory	Proof
For more information					
Thank you	!				

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