

Determinacy of Infinitely Long Games

Draft November 2020

Donald A. Martin

The main subject of this book is games in which two players are given a set A of infinite sequences of natural numbers and take turns choosing natural numbers, producing an infinite sequence. The player who moves first wins if this sequence belongs to A ; otherwise the opponent wins. Such a game is *determined* if one of the players has a winning strategy.

If A belongs to a set Γ of sets of infinite sequences of natural numbers, then we call the game a Γ game. We will present proofs of theorems of the following form: *Under hypothesis H , all Γ games are determined.* In Chapter 1, the sets Γ are the first few levels of the Borel hierarchy and the hypotheses H are the axioms of second-order arithmetic or slightly more. For most of Chapter 2, Γ is the set of all Borel sets and H is ZFC. In the remaining chapters, the sets Γ get larger and larger, and the hypotheses H are large cardinal hypotheses.

Many of these theorems have converses or quasi-converses. These are presented as exercises with hints that are essentially sketches of proofs.

The reader should have basic familiarity with set theory, but the book assumes no familiarity with games, descriptive set theory, or large cardinals.

Another section may be added later to Chapter 5. Chapters 5 and 9 still need serious proofreading. The working title of the book is close to that of Itay Neeman's book on determinacy, so it will probably be changed.

Bibliography

- [Baldwin, 1986] Baldwin, S. (1986). Between strong and superstrong. *Journal of Symbolic Logic*, 51:547–559.
- [Banach, 1930] Banach, S. (1930). Über additive Massfunktionen in abstrakten Mengen. *Fundamenta Mathematicae*, 15:97–101.
- [Blackwell, 1969] Blackwell, D. (1969). Infinite G_δ games with imperfect information. *Matematykyki Appliciones Mathematicae*, 10:99–101. Hugo Steinhaus Jubilee Volume.
- [Blass, 1975] Blass, A. (1975). Equivalence of two strong forms of determinacy. *Proceedings of the American Mathematical Society*, 52:373–376.
- [Davis, 1964] Davis, M. (1964). Infinite games of perfect information. In Dresher, M., Shapley, L. S., and Tucker, A. W., editors, *Advances in Game Theory*, Annals of Mathematics Studies #52, pages 85–101. Princeton University Press, Princeton.
- [Dodd and Jensen, 1981] Dodd, A. and Jensen, R. (1981). The core model. *Annals of Mathematical Logic*, 20:43–75.
- [Dodd,] Dodd, A. J. Strong cardinals. Unpublished notes.
- [Dodd, 1982] Dodd, A. J. (1982). *The Core Model*. London Mathematical Society Lecture Notes Series. Cambridge University Press, Cambridge.
- [Erdős and Hajnal, 1966] Erdős, P. and Hajnal, A. (1966). On a problem of b. johnson. *Bulletin de l'Académie Polonaise des Sciences Série des Sciences Mathématiques, Astronomiques, et Physiques*, 14:19–23.

- [Foreman et al., 1988] Foreman, M., Magidor, M., and Shelah, S. (1988). Martin's Maximum, saturated ideals and non-regular ultrafilters, part I. *Annals of Mathematics*, 127:1–47.
- [Friedman, 1971] Friedman, H. M. (1971). Higher set theory and mathematical practice. *Annals of Mathematical Logic*, 2:325–357.
- [Gaifman, 1964] Gaifman, H. (1964). Measurable cardinals and constructible sets (abstract). *Notices of the American Mathematical Society*, 11:771.
- [Gaifman, 1974] Gaifman, H. (1974). Elementary embeddings of models of set theory and certain subtheories. In Jech, T. J., editor, *Axiomatic Set Theory*, Proceedings of Symposia in Pure Mathematics, Volume 13, Part 2, pages 33–101, Providence. American Mathematical Society.
- [Gale and Stewart, 1953] Gale, D. and Stewart, F. (1953). Infinite games with perfect information. In Kuhn, H. and Tucker, A., editors, *Contributions to the Theory of Games, Volume 2*, Annals of Mathematics Studies #28, pages 245–266. Princeton University Press, Princeton.
- [Galvin and Prikry, 1976] Galvin, F. and Prikry, K. L. (1976). Infinitary Jonsson algebras and partition relations. *Algebra Universalis*, 6:367–376.
- [Gödel, 1939] Gödel, K. F. (1939). Consistency-proof for the generalized continuum-hypothesis. *Proceedings of the National Academy of Science USA*, 25:220–224. Reprinted in [Gödel, 1990].
- [Gödel, 1990] Gödel, K. F. (1990). *Collected Works*, volume 2. Oxford University Press, New York.
- [Hanf, 1964] Hanf, W. P. (1964). Incompactness in languages with infinitely long expressions. *Fundamenta Mathematicae*, 53:309–324.
- [Hansell, 1973a] Hansell, R. (1973a). On the nonseparable theory of k -Borel and k -Souslin sets. *General Topology and Applications*, 3:161–195.
- [Hansell, 1973b] Hansell, R. (1973b). On the representation of nonseparable analytic sets. *Proceedings of the American Mathematical Society*, 39:402–408.
- [Harrington, 1978] Harrington, L. A. (1978). Analytic determinacy and $0^\#$. *Journal of Symbolic Logic*, 43:685–693.

- [Harrington and Kechris, 1975] Harrington, L. A. and Kechris, A. S. (1975). A basis result for Σ_3^0 sets of reals with an application to minimal covers. *Proceedings of the American Mathematical Society*, 53:445–448.
- [Hurkens, 1993] Hurkens, A. J. (1993). *Borel Determinacy without the Axiom of Choice*. PhD thesis, Catholic University of Nijmegen.
- [Kanamori, 1994] Kanamori, A. (1994). *The Higher Infinite*. Springer-Verlag, Berlin.
- [Kechris, 1981] Kechris, A. S. (1981). Homogeneous trees and projective scales. In Kechris, A., Martin, D., and Moschovakis, Y., editors, *Cabal Seminar 77-79*, Lecture Notes in Mathematics #839, pages 33–73. Springer-Verlag, Berlin. Proceedings of the Caltech-UCLA Logic Seminar 1977–1979.
- [Kechris, 1994] Kechris, A. S. (1994). *Classical Descriptive Set Theory*. Springer-Verlag, New York.
- [Kechris et al., 1988] Kechris, A. S., Martin, D. A., and Steel, J. R., editors (1988). *Cabal Seminar 1981–1985*, Berlin. Proceedings of Caltech–UCLA Logic Seminar, Springer-Verlag.
- [Kechris and Woodin, 1983] Kechris, A. S. and Woodin, W. H. (1983). Equivalence of partition properties and determinacy. *Proceedings of the National Academy of Sciences USA*, 80:1783–1786.
- [Keisler, 1962a] Keisler, H. J. (1962a). The equivalence of certain problems in set theory with problems in the theory of models (abstract). *Notices of the American Mathematical Society*, 9:339.
- [Keisler, 1962b] Keisler, H. J. (1962b). Some applications of the theory of models to set theory. In Nagel, E., Suppes, P., and Tarski, A., editors, *Logic, Methodology, and Philosophy of Science*, pages 80–86, Stanford. Stanford University Press. Proceedings of the 1960 International Congress.
- [Keisler and Tarski, 1964] Keisler, H. J. and Tarski, A. (1964). From accessible to inaccessible cardinals. *Fundamenta Mathematicae*, 53:225–308.
- [Kleene, 1955] Kleene, S. C. (1955). On the form of predicates in the theory of constructive ordinals (second paper). *American Journal of Mathematics*, 77:405–428.

- [Koepke, 1998] Koepke, P. (1998). Extenders, embedding normal forms, and the Martin-Steel Theorem. *jsl*, 63:1137–1176.
- [Kunen, 1968] Kunen, K. (1968). *Inaccessibility Properties of Cardinals*. PhD thesis, Stanford University. Published in part in [Kunen, 1970].
- [Kunen, 1970] Kunen, K. (1970). Some applications of iterated ultrapowers in set theory. *Annals of Mathematical Logic*, 1:179–227.
- [Kunen, 1971] Kunen, K. (1971). Elementary embeddings and infinitary combinatorics. *Journal of Symbolic Logic*, 36:407–413.
- [Kunen, 1978] Kunen, K. (1978). Saturated ideals. *Journal of Symbolic Logic*, 43:65–76.
- [Kunen, 1980] Kunen, K. (1980). *Set Theory: An Introduction to Independence Proofs*. North-Holland, Amsterdam–New York–Oxford.
- [Kuratowski, 1958] Kuratowski, K. (1958). *Topologie*, volume 1. Państwowe Wydawnictwo Naukowe, Warsaw. English translation, 1966.
- [Łoś, 1955] Łoś, J. (1955). Quelques remarques, théorèmes et problèmes sur les classes définissables d’algèbres. In Skolem, T. et al., editors, *Mathematical Interpretation of Formal Systems*. North-Holland, Amsterdam.
- [Louveau and Saint-Raymond, 1987] Louveau, A. and Saint-Raymond, J. (1987). Borel classes and closed games: Wadge-type and Hurewicz-type results. *Transactions of the American Mathematical Society*, 304:431–467.
- [Louveau and Saint-Raymond, 1988] Louveau, A. and Saint-Raymond, J. (1988). The strength of Borel Wadge Determinacy. In [Kechris et al., 1988], pages 1–30. Proceedings of the Caltech-UCLA Logic Seminar 1981–1985.
- [Lusin, 1927] Lusin, N. N. (1927). Sur les ensembles analytiques. *Fundamenta Mathematicae*, 10:1–95.
- [Lusin and Sierpiński, 1923] Lusin, N. N. and Sierpiński, W. (1923). Sur un ensemble non mesurable B . *Journal de Mathématiques Pure et Appliquées*, 9^e série, 2:53–72.

- [Maitra et al., 1991] Maitra, A., Purves, R., and Sudderth, W. (1991). Approximation theorems for gambling problems and stochastic games. In Dutta, B., Mookherjee, D., Parthasarathy, T., Raghavan, T., Ray, D., and Tijs, S., editors, *Game Theory and Economic Applications*, pages 114–132. Springer-Verlag, Berlin. Proceedings of conference in New Delhi, December 1990.
- [Maitra and Sudderth, 1992] Maitra, A. and Sudderth, W. (1992). An operator solution of stochastic games. *Israel Journal of Mathematics*, 78:33–49.
- [Maitra and Sudderth, 1993] Maitra, A. and Sudderth, W. (1993). Finitely additive and measurable stochastic games.
- [Martin, 1968] Martin, D. A. (1968). The axiom of determinateness and reduction principles in the analytical hierarchy. *Bulletin of the American Mathematical Society*, 74:687–68.
- [Martin, 1970] Martin, D. A. (1970). Measurable cardinals and analytic games. *Fundamenta Mathematicae*, 66:287–291.
- [Martin, 1975] Martin, D. A. (1975). Borel determinacy. *Annals of Mathematics*, 102:363–371.
- [Martin, 1980] Martin, D. A. (1980). Infinite games. In Lehto, O., editor, *Proceedings of the International Congress of Mathematicians, Helsinki 1978*, volume 1, pages 269–273, Helsinki. Academia Scientiarum Fennica.
- [Martin, 1985] Martin, D. A. (1985). A purely inductive proof of Borel determinacy. In Nerode, A. and Shore, R. A., editors, *Recursion Theory*, Proceedings of Symposia in Pure Mathematics, Volume 42, pages 303–308, Providence. American Mathematical Society.
- [Martin, 1990] Martin, D. A. (1990). An extension of Borel determinacy. *Annals of Pure and Applied Logic*, 49:279–293.
- [Martin, 1998] Martin, D. A. (1998). The determinacy of Blackwell games. *Journal of Symbolic Logic*, 63:1565–1581.
- [Martin, 2003] Martin, D. A. (2003). A simple proof that determinacy implies Lebesgue measurability. *Rendiconti Del Seminario Matematico, Università e Politecnico di Torino*, 61:393–398.

- [Martin, 2015] Martin, D. A. (2015). Games of countable length. In Kechris, A., Loewe, B., and Steel, J., editors, *The Cabal Seminar, Volume IV*, Cambridge, New York, etc. Cambridge University Press.
- [Martin et al., 2003] Martin, D. A., Neeman, I., and Vervoort, M. R. (2003). The strength of Blackwell determinacy. *Journal of Symbolic Logic*, 68:615–636.
- [Martin and Solovay, 1969] Martin, D. A. and Solovay, R. M. (1969). A basis theorem for Σ_3^1 sets of reals. *Annals of Mathematics*, 89:138–159.
- [Martin and Solovay, 1970] Martin, D. A. and Solovay, R. M. (1970). Internal Cohen extensions. *Annals of Mathematical Logic*, 2:143–178.
- [Martin and Steel, 1988] Martin, D. A. and Steel, J. R. (1988). Projective determinacy. *Proceedings of the National Academy of Science USA*, 85:6582–6586.
- [Martin and Steel, 1989] Martin, D. A. and Steel, J. R. (1989). A proof of projective determinacy. *Journal of the American Mathematical Society*, 2:71–125.
- [Martin and Steel, 1994] Martin, D. A. and Steel, J. R. (1994). Iteration trees. *Journal of the American Mathematical Society*, 7:1–73.
- [Mauldin, 1981] Mauldin, R. D., editor (1981). *The Scottish Book: Mathematics from the Scottish Café*. Birkhäuser, Boston.
- [Mitchell, 1974] Mitchell, W. (1974). Sets constructible from sequences of ultrafilters. *Journal of Symbolic Logic*, 39:57–66.
- [Mitchell, 1979] Mitchell, W. (1979). Hypermeasurable cardinals. In Boffa, M., Dalen, D. V., and McAloon, K., editors, *Logic Colloquium '78*, pages 303–317, Amsterdam. North Holland.
- [Montalban and Shore,] Montalban, A. and Shore, R. The limits of determinacy in second order arithmetic: Consistency and complexity strength. *Israel Journal of Mathematics*. To appear.
- [Montalban and Shore, 2012] Montalban, A. and Shore, R. (2012). The limits of determinacy in second order arithmetic. *Proceedings of the London Mathematical Society*, 104:223–252.

- [Moschovakis, 1980] Moschovakis, Y. N. (1980). *Descriptive Set Theory*. North-Holland, Amsterdam.
- [Moschovakis, 2009] Moschovakis, Y. N. (2009). *Descriptive Set Theory, Second Edition*. American Mathematical Society, Providence.
- [Mostowski, 1949] Mostowski, A. M. (1949). An undecidable arithmetical statement. *Fundamenta Mathematicae*, 36:143–164.
- [Mycielski, 1964] Mycielski, J. (1964). On the axiom of determinateness. *Fundamenta Mathematicae*, 53:205–224.
- [Mycielski and Steinhaus, 1962] Mycielski, J. and Steinhaus, H. (1962). A mathematical axiom contradicting the axiom of choice. *Bulletin de l'Académie Polonaise des Sciences Série des Sciences Mathématiques, Astronomiques, et Physiques*, 10:1–3.
- [Neeman, 1995] Neeman, I. (1995). Optimal proofs of determinacy. *Bulletin of Symbolic Logic*, 1:327–339.
- [Neeman, 2000] Neeman, I. (2000). Unraveling Π_1^1 sets. *Annals of Pure and Applied Logic*, 105:151–205.
- [Neeman, 2002] Neeman, I. (2002). Optimal proofs of determinacy II. *Journal of Mathematical Logic*, 2:227–258.
- [Neeman, 2004] Neeman, I. (2004). *The Determinacy of Long Games*. Walter de Gruyter and Co., Berlin.
- [Neeman, 2007] Neeman, I. (2007). Optimal proofs of determinacy ii. *Journal of Mathematical Logic*, 7:83–124.
- [Neeman, 2010] Neeman, I. (2010). Determinacy in $L(\mathbb{R})$. In Kanamori, A. and Foreman, M., editors, *Handbook of Set Theory*, volume 3, pages 1887–1950. Springer-Verlag, Berlin.
- [Neeman and Steel, 2006] Neeman, I. and Steel, J. (2006). Counterexamples to the Unique and Cofinal Branches Hypotheses. *Journal of Symbolic Logic*, 71:977–988.
- [Orkin, 1972] Orkin, M. (1972). Infinite games with imperfect information. *Transactions of the American Mathematical Society*, 171:501–507.

- [Oxtoby, 1957] Oxtoby, J. C. (1957). The Banach–Mazur game and the Banach Category theorem. In Dresner, M., Tucker, A. W., and Wolfe, P., editors, *Contributions to the Theory of Games, Volume 3*, Annals of Mathematics Studies #29, pages 159–163. Princeton University Press, Princeton.
- [Paris, 1972] Paris, J. B. (1972). $ZF \vdash \Sigma_4^0$ determinateness. *Journal of Symbolic Logic*, 37:661–667.
- [Rowbottom, 1964] Rowbottom, F. (1964). *Large Cardinals and Small Constructible Sets*. PhD thesis, University of Wisconsin at Madison. Published as [Rowbottom, 1971].
- [Rowbottom, 1971] Rowbottom, F. (1971). Some strong axioms of infinity incompatible with the axiom of constructibility. *Annals of Mathematical Logic*, 3:1–44.
- [Sargsyan and Trang, 2016] Sargsyan, G. and Trang, N. (2016). Tame failures of the unique branch hypothesis and models of $AD_{\mathbb{R}} + \theta$ is regular. *Journal of Mathematical Logic*, 16:??
- [Scott, 1955] Scott, D. S. (1955). Definitions by abstraction in axiomatic set theory. *Bulletin of the American Mathematical Society*, 61:442.
- [Scott, 1961] Scott, D. S. (1961). Measurable cardinals and constructible sets. *Bulletin de l'Académie Polonaise des Sciences Série des Sciences Mathématiques, Astronomiques, et Physiques*, 9:521–524.
- [Shelah and Woodin, 1990] Shelah, S. and Woodin, W. H. (1990). Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable. *Israel Journal of Mathematics*, 70:381–394.
- [Silver, 1966] Silver, J. H. (1966). *Some Applications of Model Theory in Set Theory*. PhD thesis, University of California at Berkeley. Published in abridged form as [Silver, 1971].
- [Silver, 1971] Silver, J. H. (1971). Some applications of model theory in set theory. *annmathlog*, 3:45–110.
- [Solovay, 1967] Solovay, R. M. (1967). A nonconstructible Δ_3^1 set of integers. *Transactions of the American Mathematical Society*, 127:50–75.

- [Solovay et al., 1978] Solovay, R. M., Reinhardt, W. N., and Kanamori, A. (1978). Strong axioms of infinity and elementary embeddings. *Annals of Mathematical Logic*, 13:73–116.
- [Steel, 1976] Steel, J. R. (1976). *Forcing with Tagged Trees*. PhD thesis, University of California at Berkeley.
- [Steel, 2002] Steel, J. R. (2002). Core models with more woodin cardinals. *Journal of Symbolic Logic*, 67:1197–1226.
- [Tarski, 1962] Tarski, A. (1962). Some problems and results relevant to the foundations of set theory. In Nagel, E., Suppes, P., and Tarski, A., editors, *Logic, Methodology, and the Philosophy of Science*, volume 1 of *Proceedings of the 1960 International Congress*, pages 125–135, Stanford. Stanford University Press.
- [Ulam, 1930] Ulam, S. M. (1930). Zur Masstheorie in der allgemeinen Mengenlehre. *Fundamenta Mathematicae*, 14:140–150.
- [Vervoort, 1996] Vervoort, M. R. (1996). Blackwell games. In Ferguson, T., Shapley, L., and MacQueen, J., editors, *Statistics, Probability and Game Theory: Papers in honor of David Blackwell*, Lecture Notes—Monograph Series, pages 369–390. Institute of Mathematical Statistics, Hayward.
- [Vervoort, 2000] Vervoort, M. R. (2000). *Games, Walks, and Grammars: Problems I've Worked on*. PhD thesis, Universiteit van Amsterdam.
- [von Neumann, 1928] von Neumann, J. (1928). Zur Theorie der Gesellschaftsspiele. *Mathematische Annalen*, 100:295–320.
- [Wolfe, 1955] Wolfe, P. (1955). The strict determinateness of certain infinite games. *Pacific Journal of Mathematics*, 5:841–847.
- [Zermelo, 1913] Zermelo, E. (1913). Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels. In Hobson, E. and Love, A., editors, *Proceedings of the Fifth International Congress of Mathematicians, Cambridge, 1912*, volume 2, pages 501–504, Cambridge. Cambridge University Press.