Stationary particle Systems

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Poisson process

Definition

Let Λ be a Radon measure on $\mathbb{R}^p$. A *Poisson process* Π is a random point measure on $\mathbb{R}^d$ satisfying the following two conditions.

(i) $\Pi(A) \sim Po(\Lambda(A))$ for every bounded $A \in \mathcal{B}(\mathbb{R}^p)$.

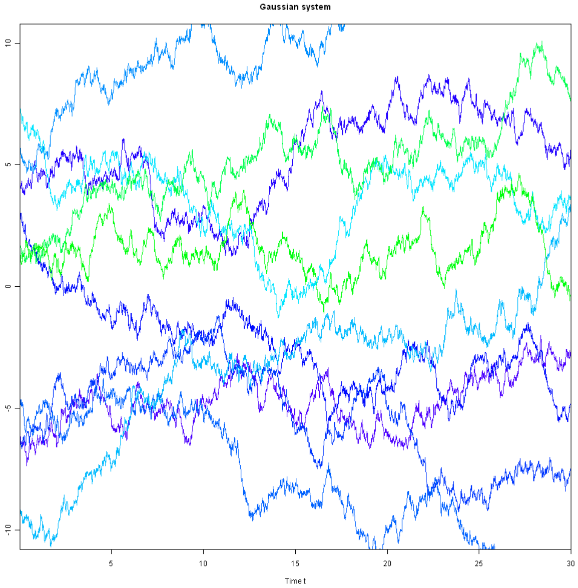
(ii) For all bounded and disjoint sets $A_1, A_2 \in \mathcal{B}(\mathbb{R}^p)$ the random variables $\Pi(A_1), \Pi(A_2)$ are independent.

We identify the Poisson process with its points, i.e. $\Pi = \{x_i, i \geq 1\}$.

particle System

- Let $\Pi = \sum_{i=1}^{M} \delta_{x_i}$ be a Poisson process on $\mathbb{R}^p$ with intensity measure Λ, and let $\{\xi_i(t)\}_{t\in\mathbb{R}^d}$ be i.i.d stochastic processes independent of Π.
- $\{\Pi_t\}_{t\in\mathbb{R}^d} = \{x_i + \xi_i(t),\ i \in \mathbb{N}\}_{t\in\mathbb{R}^d}$ is called *independent particle system* (or simply *particle system*) generated by the pair (Λ, ξ).
- In order that Π_t is well-defined, Λ and ξ have to fulfil certain (integrability) conditions.
- Our goal is to describe particle systems, which are stationary.

Example. $\Lambda(dx) = dx$, $\xi(t) = W(t)$ (Wiener process)



Stationarity

The particle system generated by (Λ, ξ) is stationary, if and only if the "finite dimensional versions"

$$\Pi_{t_1,\dots,t_n} = \{x_i + \xi_i(t_1), \dots, x_i + \xi_i(t_n)\,,\, i \in \mathbb{N}\}$$

are invariant under time shifts, i.e. for all $h \in \mathbb{R}^d$

$$\Pi_t \sim \Pi_{t+h}$$
$$\Pi_{t_1,t_2} \sim \Pi_{t_1+h,t_2+h}$$
$$\vdots$$

Proposition

The point process $\Pi_{t_1,\ldots,t_n}$ *is a Poisson process on the space* $\mathbb{R}^{pn}$ *with intensity measure*

$$\Lambda_{t_1,\ldots,t_n}(A) = \int_{\mathbb{R}^p} \mathbf{P}\left((x+\xi(t_1),\ldots,x+\xi(t_n)) \in A\right)\Lambda(dx)\,.$$

The right hand side is the convolution of $P_{\xi(t_1),\ldots,\xi(t_n)}$ and the product $\Lambda \otimes \delta_{x_2=x_1} \otimes \cdots \otimes \delta_{x_n=x_1}$.
All convolutions are locally finite measures, if $P_{\xi(t)} * \Lambda$ is locally finite for all $t \in \mathbb{R}^d$.

Since two Poisson processes are equal if and only if their intensity measures are equal, the following system of convolution equations must hold for all $h, t_1, ..., t_n \in \mathbb{R}^d$.

$$\Lambda_t = \Lambda_{t+h}\,, \quad \text{i.e.} \quad P_{\xi(t)} * \Lambda = P_{\xi(t+h)} * \Lambda\,,$$

and further equations $\Lambda_{t_1,...,t_n} = \Lambda_{t_1+h,...,t_n+h}$.
Unfortunately, there is *no general theory* describing all solutions of such a convolution equation. However if it can be transformed in a *one-sided equation*, there is hope to solve it.

If $P_{\xi(t_1)}$ and $P_{\xi(t_2)}$ are univariate Gaussian measures, then its possible to "substract" them and transform two-sided equation $\Lambda * P_{\xi(t_1)} = \Lambda * P_{\xi(t_2)}$ to the one-sided equation

$$\Lambda * P = \Lambda\,.$$

Dény equation

Theorem (Dény 1960)

Let P a probability measure with support $\mathbb{R}^d$, then the solution of $\Lambda * P = \Lambda$ has the density

$$\frac{\Lambda(dx)}{dx} = f_\Lambda(x) = \int_{E(P)} e^{-\langle \lambda, x\rangle} Q(d\lambda)\,,$$

where Q is a measure concentrated on the set

$$E(P) = \left\{ \lambda \in \mathbb{R}^d : \int_{\mathbb{R}^d} e^{\langle \lambda, x\rangle} P(dx) = 1 \right\}.$$

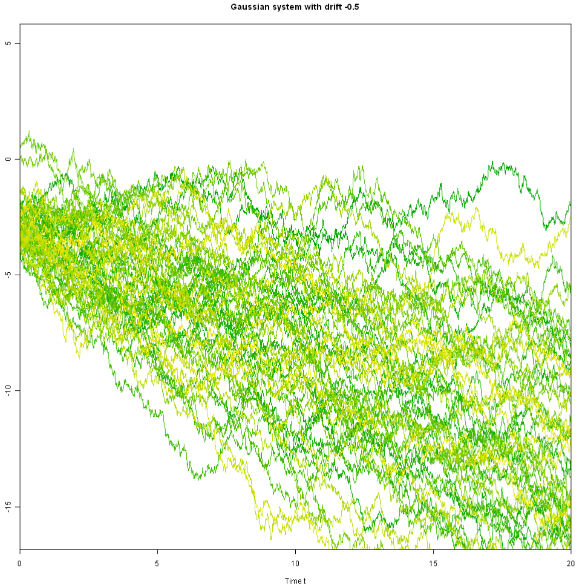
Classification of univariate Gaussian systems

Theorem (Kabluchko 2010)

Let (Λ, ξ) *be a stationary Gaussian systems. Then either*

- Λ *is an arbitrary measure and* ξ *is a stationary Gaussian process.*
- Λ *is proportional to the Lebesgue measure and* $\xi(t) = W(t) + f(t) + c$*, where* W *is a Gaussian process with zero mean and stationary increments and* $f(t)$ *is an additive function.*
- Λ *has the density* $f_\Lambda(x) = \alpha e^{-\lambda x}$ *and* $\xi(t) = W(t) - \lambda\sigma_t^2/2 + c$*, where* $W(t)$ *is a Gaussian process with zero mean, stationary increments and variance* σ_t^2*.*

Ex. Brown-Resnick $\Lambda(dx) = e^{-x}dx$, $\xi(t) = W(t) - 1/2t$



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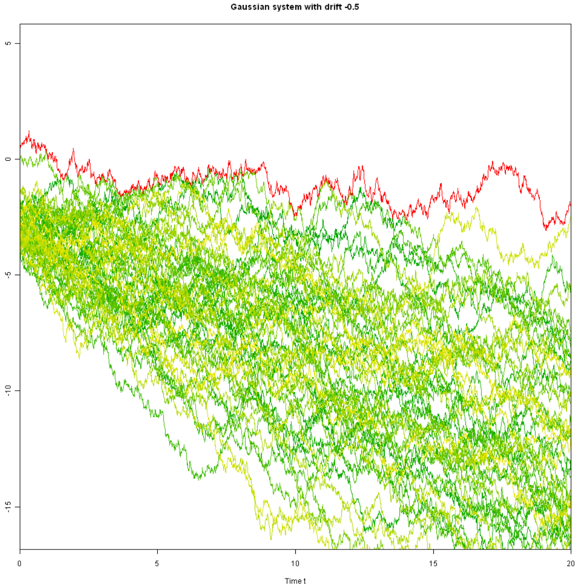

New apporach: Spectral synthesis

Assume that Λ has a density f_Λ. The convolution equation can be written as

$$f_\Lambda * (P_{\xi(t_1)} - P_{\xi(t_2)}) = f_\Lambda * \mu = 0\,.$$

Definition

A continuous function f is called *mean-periodic* if there exists a signed measure μ with compact support, such that $\mu * f = 0$ (f is μ-mean-periodic).

Theorem (Spectral synthesis theorem (Schwartz, 1947))

In the univariate case, the linear hull of all exponential monomials ($x^p e^{-\lambda x}$) is dense in the set of μ-mean-periodic functions.

Unfortunately, this is wrong for $p \geq 2$. And there is also no theory for unbounded μ.

Multivariate Gaussian particle systems

- "Subtraction" of $P_{\xi(t_1)}$ and $P_{\xi(t_2)}$ is possible in the univariate Gaussian case.
- But if $p \geq 2$ its no longer possible, as in general the difference of two covariance matrices is neither positive nor negative definite.
- However, a great class of solutions is of the form $\int_E e^{-\langle \lambda, x \rangle} Q(d\lambda)$, where Q is concentrated on the set

$$E = \left\{ \lambda : \mathbf{E}\left(e^{\langle \lambda, \xi(t_1) \rangle} \right) = \mathbf{E}\left(e^{\langle \lambda, \xi(t_2) \rangle} \right) \right\}$$

- But are these all solutions?

The situation resembles a bit to the theory of second order PDE's. They are classified into elliptic, parabolic and hyperbolic PDE's according to its characteristic polynomial. The set E is characterised through a quadratic polynomial and if its hyperbolic, "strange"solutions may occur.

Example (Not a exponential measure)

Let ξ_1, ξ_2 be two bivariate normal distributions

$$\xi_1 \sim \mathcal{N}_2(0, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}) \quad \text{and} \quad \xi_2 \sim \mathcal{N}_2(0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}) .$$

Then $\Lambda * P_{\xi_1} = \Lambda * P_{\xi_2}$ holds for every measure Λ with density $f_\Lambda(x_1, x_2) = g(x_1 + x_2)$, for a arbitrary function g satisfying a suitable integrable condition.

Question: Does there exists a Gaussian process, such that the particle system is stationary and such that all characteristic polynomials are hyperbolic?

Λ has a exponential polynomial density

Theorem

Assume that for all $t_1, \cdots, t_n \in \mathbb{R}^d$ the probability measure $P_{(\xi(t_1),\ldots,\xi(t_n))}$ has the density $f_{t_1,\ldots,t_n}$. The particle system $\Pi(t)$ generated by Λ with density

$$x \mapsto e^{-\langle \lambda, x\rangle} \sum_{|\alpha|\leq k} c_\alpha x^\alpha$$

is stationary if and only if for all multi indices $\beta \leq \alpha$ the particle systems generated by intensity measures with densities $x \mapsto e^{-\langle \lambda, x\rangle} x^\beta$ are stationary

Let $\Lambda = e_\lambda$, where e_λ denotes the exponential measure with density $f(x) = e^{-\langle \lambda, x \rangle}$. Analogue to the one-dimensional case we can show

Theorem

The Gaussian system $GS(e_\lambda, \xi(t))$ is stationary if and only if the process $\xi(t)$ is of the form

$$\xi(t) = W_t - \frac{1}{2}\Sigma_{t,t}\lambda + b_t + c\,, \tag{1}$$

where W_t is a Gaussian process with zero mean, variance $\Sigma_{t,t}$ and stationary increments, b_t is an additive function orthogonal to λ and c is a constant.

Mixture of exponential measures

Assume the measure Λ has the density $f_\Lambda(x) = \int_E e^{-\langle \lambda, x \rangle} dQ(\lambda)$.

Theorem

The Gaussian system $GS(\Lambda, \xi_t)$ is stationary, if and only if for all λ in the support of Q, the system $GS(e_\lambda, \xi_t)$ is stationary.

Lemma

Let $\lambda_1, \lambda_2 \in \mathbb{R}^d$, $\lambda_1 \neq \lambda_2$. If the Gaussian systems $GS(e_{\lambda_1}, \xi(t))$ and $GS(e_{\lambda_2}, \xi(t))$ are both stationary, then the one-dimensional process $\xi_{\Delta\lambda}(t)$, $t \in \mathbb{R}$ given by

$$\xi_{\Delta\lambda}(t) = \langle \xi(t), \Delta\lambda \rangle , \quad \Delta\lambda = \lambda_1 - \lambda_2$$

is stationary.

Conclusion

- It seems not to be possible to solve the system of convolution equation analytically, but it may be possible probabilistically, i.e. using properties of the hole process, e.g. ergodic properties.
- If Λ is a mixture of exponential measures, e.g. assuming $\xi(0) = 0$, and additionally ξ is in no direction stationary, then we have a analogous result as in the univariate case.
- It is possible to describe stationary systems with a non-Gaussian process ξ. For instance the case with Lévy processes is well understood.

Jaques Deny.
Sur l'equation de convolution $\mu = \mu * \sigma$.
Seminaire Brelot-Choquet-Deny. Theorie du potentiel, tome 4:1–11, (1959-1960).

Zakhar Kabluchko.
Stationary systems of Gaussian processes.
Ann. Appl. Probab., 20(6):2295–2317, 2010.