Unitary Matrix Models: universality conjecture in the bulk and on the edge of the spectrum.

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Some types of Random Matrix Ensembles

- Matrix Ensembles with independent entries
  - Wigner matrices

$$P_n(dM) = \prod_{1 \le i < j \le n} F(n^{1/2} dM_{i,j}) \prod_{i=1}^{n} F_0(n^{1/2} dM_{i,i})$$

  - Marchenko-Pastur ensemble

$$P_n(dA) = \prod_{1 \le n, j \le m} F(n^{1/2} dA_{i,j}), \quad M_n = n^{-1} A A^*$$

- Hermitian and Unitary Matrix Ensembles
  - Hermitian and Real Symmetric Matrix Models

$$P_{n,\beta}(d_\beta M) = Z_{n,\beta}^{-1} exp\left\{ -\frac{\beta n}{2} \mathrm{Tr}\ V(M) \right\} d_\beta M.$$

  - Unitary Matrix Models

$$p_n(U)\, d\mu_n(U) = Z_n^{-1} exp\left\{ -nTrV\left(\frac{U + U^*}{2} \right) \right\} d\mu_n(U).$$

Joint Eigenvalue Distribution

Let Let $\left\{e^{i\lambda_j^{(n)}}\right\}_{j=1}^{n}$ be an eigenvalues of matrix U.

$$p_n(\lambda_1,\ldots,\lambda_n)=\frac{1}{Z_n}\prod_{1\le j<k\le n}\left|e^{i\lambda_j}-e^{i\lambda_k}\right|^2 e^{-n\sum\limits_{j=1}^{n}V(\cos\lambda_j)}.$$

$$p_l^{(n)}(\lambda_1,\ldots,\lambda_l)=\int p_n(\lambda_1,\ldots,\lambda_l,\lambda_{l+1},\ldots,\lambda_n)\,d\lambda_{l+1}\ldots d\lambda_n.$$

OPUC with a varying weight and determinant formulas

$$\left\{e^{ik\lambda}\right\}_{k=0}^{n}\to P_k^{(n)}\left(e^{i\lambda}\right):\int P_k^{(n)}\left(e^{i\lambda}\right)\overline{P_l^{(n)}\left(e^{i\lambda}\right)}e^{-nV(\cos\lambda)}d\lambda=\delta_{k,l}$$

$$K_n\left(e^{i\lambda},e^{i\mu}\right)=\sum_{j=0}^{n-1}P_j^{(n)}\left(e^{i\lambda}\right)\overline{P_j^{(n)}\left(e^{i\mu}\right)}e^{-nV(\cos\lambda)/2}e^{-nV(\cos\mu)/2}$$

$$p_l^{(n)}(\lambda_1,\ldots,\lambda_l)=\frac{(n-l)!}{n!}det\left\|K_n\left(e^{i\lambda_j},e^{i\lambda_k}\right)\right\|_{j,k=1}^{l}$$

Global and local regimes

- Global regime: $N_n(\Delta) = n^{-1}\sharp\left\{\lambda_j^{(n)} \in \Delta,\, l = 1, \ldots, n\right\},\ \Delta \in [-\pi, \pi)$

$$N_n(\Delta) = \int_\Delta p_1^{(n)}(\lambda)\, d\lambda \overset{?}{\to} N(\Delta) = \int_\Delta \rho(\lambda)\, d\lambda,\ n \to \infty.$$

- Local regime:

$$[c_V \delta_n]^{-l} p_l^{(n)}\left(\overrightarrow{\Lambda_0} + \frac{\overrightarrow{\xi}}{c_V \delta_n}\right) \overset{?}{\to} \det\{K(\xi_j, \xi_k)\}_{j,k=1}^{l}.$$

δ_n is a typical distance between eigenvalues $\Rightarrow \int_{|\lambda-\lambda_0|\le\delta_n} \rho(\lambda)d\lambda \sim \frac{1}{n}$.

  - Bulk universality: $\rho(\lambda_0) \neq 0 \Rightarrow \delta_n = n^{-1}$.
  - Edge universality: $\rho(\lambda) \sim |\lambda - \lambda_0|^{1/2} \Rightarrow \delta_n = n^{-2/3}$.

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References

- L. Pastur, M. Shcherbina '97, '07 - proved bulk and edge universality for HMM.
- A. Kolyandr '97 - studied the global regime for UMM.
- K. Johansson '98 - studied the question about length of longest increasing subsequence.
- P. Deift and colaborators '99,'99- proved uniform assymptotics for OPRL with a varying weight.
- M.J. Cantero, L. Moral, L. Velasquez '03 - obtained the five term recurrence relation for OPUC.
- K. T.-R. McLaughlin '06- proved assymptotics for OPUC $(\rho(\lambda) > 0)$.

Global regime

The joint eigenvalue distribution can be rewritten in terms of Hamiltonian $p_n(\Lambda) = \frac{1}{Z_n} e^{-nH(\Lambda)}$ with

$$H(\Lambda) = \sum_{j=1}^{n} V(\cos\lambda_j) - \frac{2}{n} \sum_{1 \le j < k \le n} \log\left|e^{i\lambda_j} - e^{i\lambda_k}\right|.$$

Consider the linear functional

$$\mathcal{E}[m] = \int_{-\pi}^{\pi} V(\cos\lambda) m(d\lambda) - \int_{-\pi}^{\pi} \log\left|e^{i\lambda} - e^{i\mu}\right| m(d\lambda) m(d\mu),$$

in the class of unit measures on the interval $[-\pi, \pi]$.

Theorem

Let potential $V(\cos\lambda)$ be a $C^2[-\pi,\pi]$, then there exists a unique minimizer of the functional,called an equilibrium measure. This measure has a density $\rho(\lambda)$ and NCM measure of eigenvalues converges in probability to the equilibrium measure.

Bulk universality

Theorem

Let potential $V(\cos\lambda)$ *be a* $C^2[-\pi,\pi]$ *function and there exists some subinterval* $(a,b) \subset \operatorname{supp} \rho(\lambda)$ *such that* $\sup\limits_{\lambda\in(a,b)} V'''(\lambda) \le C_1,\ \rho(\lambda) \ge C_2,\ \lambda \in (a,b)$. *Then the universality conjecture is true for every* $\lambda_0 \in (a,b)$ *with kernel* $K(x,y) = \dfrac{\sin\pi(x-y)}{\pi(x-y)}$ *and* $c_V = \rho(\lambda_0)$.*The limit is uniform for any* $\vec{\xi}$ *in a compact subset of* $\mathbb{R}^l$.

Basic ideas of the proof

- Prove the uniform convergence of $\rho_n(\lambda)$ to $\rho(\lambda)$.
- Derive the integro-differential equation for the K_n.
- Find the class of functions in which this equation has a unique solution.

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Basic assumptions

Condition C1. *The support* σ *of the equilibrium measure is a single subinterval of the interval* $[-\pi, \pi]$, *i.e.*

$$\sigma = [-\theta, \theta]\,, \text{ with } \quad \theta < \pi. \tag{1}$$

Condition C2. *The equilibrium density* ρ *has no zeros in* $(-\theta, \theta)$ *and*

$$\rho(\lambda) \sim C\,|\lambda \mp \theta|^{1/2}\,, \text{ for } \lambda \to \pm\theta, \tag{2}$$

and the function $u(\lambda) = V(\cos\lambda) - 2\int_\sigma \log\left|e^{i\lambda} - e^{i\mu}\right| \rho(\mu)\,d\mu$ *attains its minimum if and only if* λ *belongs to* σ.

Condition C3. $V(\cos\lambda)$ *possesses 4 bounded derivatives on* $\sigma_\varepsilon = [-\theta - \varepsilon, \theta + \varepsilon]$.

Propposition

Under conditions C1-C3

$$\rho(\lambda) = \frac{1}{4\pi^2}\chi(\lambda)\,P(\lambda)\,\mathbf{1}_\sigma, \quad \text{with}$$

$$\chi(\lambda) = \sqrt{|\cos\lambda - \cos\theta|}, \quad P(\lambda) = \int_{-\theta}^{\theta} \frac{(V(\cos\mu))' - (V(\cos\lambda))'}{\sin(\mu - \lambda)/2}\,\frac{d\mu}{\chi(\mu)}.$$

It follows from Szegö's condition that the system $\left\{P_k^{(n)}\left(e^{i\lambda}\right)\right\}_{k=0}^{\infty}$ is not complete. Following Cantero-Moral-Velasquez we define reversed polynomials $Q_k^{(n)}(\lambda) = e^{ik\lambda}P_k^{(n)}\left(e^{-i\lambda}\right)$ and Laurent polynomials

$$\begin{aligned} \chi_{2k}^{(n)}(\lambda) &= e^{-ik\lambda}Q_{2k}^{(n)}\left(e^{i\lambda}\right), \\ \chi_{2k+1}^{(n)}(\lambda) &= e^{-ik\lambda}P_{2k+1}^{(n)}\left(e^{i\lambda}\right). \end{aligned}$$

$$\begin{aligned} e^{i\lambda}\chi_{2k-1}^{(n)}(\lambda) = &-\alpha_{2k}^{(n)}\rho_{2k-1}^{(n)}\chi_{2k-2}^{(n)}(\lambda) - \alpha_{2k}^{(n)}\alpha_{2k-1}^{(n)}\chi_{2k-1}^{(n)}(\lambda) \\ &-\alpha_{2k+1}^{(n)}\rho_{2k}^{(n)}\chi_{2k}^{(n)}(\lambda) + \rho_{2k}^{(n)}\rho_{2k+1}^{(n)}\chi_{2k+1}^{(n)}(\lambda), \\ e^{i\lambda}\chi_{2k}^{(n)}(\lambda) = &\ \rho_{2k}^{(n)}\rho_{2k-1}^{(n)}\chi_{2k-2}^{(n)}(\lambda) + \alpha_{2k-1}^{(n)}\rho_{2k}^{(n)}\chi_{2k-1}^{(n)}(\lambda) \\ &-\alpha_{2k+1}^{(n)}\alpha_{2k}^{(n)}\chi_{2k}^{(n)}(\lambda) + \alpha_{2k}^{(n)}\rho_{2k+1}^{(n)}\chi_{2k+1}^{(n)}(\lambda), \end{aligned}$$

where $\alpha_k^{(n)} = c_{k,0}^{(n)}/c_{k,k}^{(n)}$ and $\rho_k^{(n)} = c_{k-1,k-1}^{(n)}/c_{k,k}^{(n)}$ are called the Verblunsky coefficients and $\left(\rho_k^{(n)}\right)^2 + \left(\alpha_k^{(n)}\right)^2 = 1$.

Assymptotics of Verblunsky coefficients

Theorem

Consider the system of orthogonal polynomials and the Verblunsky coefficients defined above. Let potential V satisfy conditions C1 - C3 above. Then for any $|m| = \overline{o}(n)$

$$\alpha_{n+m}^{(n)} = (-1)^m s \cos\left(\frac{\theta}{2} + x_m^{(n)}\right),$$

where $s = 1$ *or* $s = -1$ *and*

$$x_m = \frac{2\pi\sqrt{2}}{P(\theta)\sin\theta}\frac{m}{n} + \underline{O}\left(\log^{11} n\left(n^{-4/3} + \frac{m^2}{n^2}\right)\right),$$

with P defined above.

Basic ideas of the proof

- Derive an equation with a functional parameter ϕ for functions $\psi_k^{(n)} = P_k^{(n)} e^{-nV/2}$ from the determinant formulas. Then, choosing appropriate parameter ϕ, obtain the equation for the Verblunsky coefficients. In this way we obtain a first approximation for Verblunsky coefficients.
- Using "string" equation

$$\int_{-\pi}^{\pi} (\sin\lambda)\, V'(\cos\lambda)\, \chi_k^{(n)}(\lambda)\, \overline{\chi_{k-1}^{(n)}(\lambda)} e^{-nV(\cos\lambda)} d\lambda = i(-1)^{k-1} \frac{k}{n} \frac{\alpha_k^{(n)}}{\rho_k^{(n)}}.$$

and methods of the perturbation theory obtain assymptotics described above.

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CMV matrices

$$\overrightarrow{\chi^{(n)}} = \left\{\chi_k^{(n)}\right\}_{k=0}^{\infty}, \quad \overrightarrow{\widehat{\chi}^{(n)}} = \left\{\widehat{\chi}_k^{(n)}\right\}_{k=0}^{\infty}.$$

$$\Theta_j^{(n)} = \begin{pmatrix} -\alpha_j^{(n)} & \rho_j^{(n)} \\ \rho_j^{(n)} & \alpha_j^{(n)} \end{pmatrix},$$

$$M^{(n)} = \mathrm{diag}\left(E_1, \Theta_2^{(n)}, \Theta_4^{(n)}..\right), \quad L^{(n)} = \mathrm{diag}\left(\Theta_1^{(n)}, \Theta_3^{(n)}, \Theta_5^{(n)}..\right),$$

$$C^{(n)} = M^{(n)} L^{(n)}$$

$$\overrightarrow{\widehat{\chi}^{(n)}} = M^{(n)} \overrightarrow{\chi^{(n)}}, \quad e^{i\lambda} \overrightarrow{\widehat{\chi}^{(n)}} = L^{(n)} \overrightarrow{\chi^{(n)}}, \quad e^{i\lambda} \overrightarrow{\chi^{(n)}} = C^{(n)} \overrightarrow{\chi^{(n)}}.$$

Our main idea is to study the kernel K_n near the edge. For this aim we consider the integral operator

$$F_n^{(n)}(z, w) = \int w_n(\lambda)\, d\lambda \int w_n(\mu)\, d\mu G_{\lambda,z} G_{\mu,w} \left| \left(e^{i\lambda} - e^{i\mu}\right) K_n^{(n)}(\lambda, \mu) \right|^2,$$

where

$$G_{\lambda,z} = \frac{1 - e^{i(z-\bar{z})}}{|e^{i\lambda} - e^{iz}|^2} = e^{iz} \frac{1}{e^{i\lambda} - e^{iz}} - e^{i\bar{z}} \frac{1}{e^{i\lambda} - e^{i\bar{z}}}.$$

Edge universality

Theorem

Under assumptions C1-C3 the universality conjecture is true for $\lambda_0 = \pm\theta$ *with kernel* $K(x,y) = \dfrac{Ai(x)\,Ai'(y) - Ai'(x)\,Ai(y)}{x-y}$. *The limit is uniform for any* $\overrightarrow{\xi}$ *in a compact subset of* $\mathbb{R}^l$.

Basic ideas of the proof

- Christoffel-Darboux formula + spectral theory give us a representation of F_n in terms of resolvent of matrix $C^{(n)}$ (five-diagonal).
- Relation between matrices $C^{(n)}$, $M^{(n)}$, and $L^{(n)}$ reduces this representation to the question about resolvent of the three diagonal matrix.
- Assymptotics of Verblunsky coefficients help us to "guess" resolvent for $z = \pm\theta + n^{-2/3}\zeta$. It can be represented in terms of resolvent $(A-\zeta)^{-1}$ of operator $A = \dfrac{d^2}{dx^2} - 2cx$.

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