Universality and RSW for inhomogeneous bond percolation

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joint work with Geoffrey Grimmett

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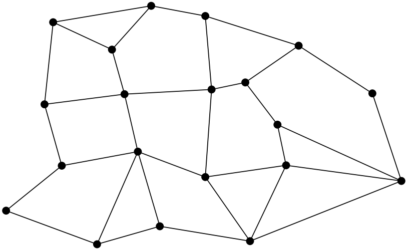
22 August 2011

Percolation



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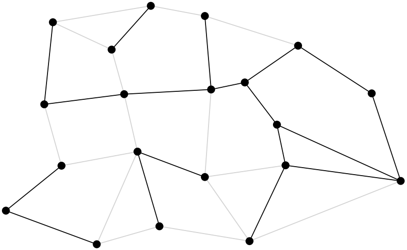

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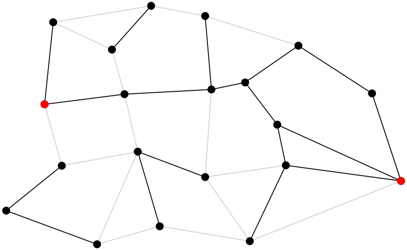

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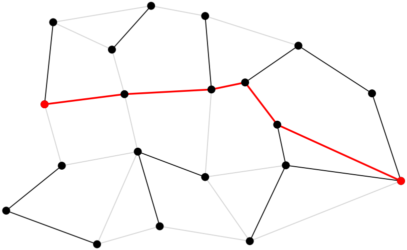

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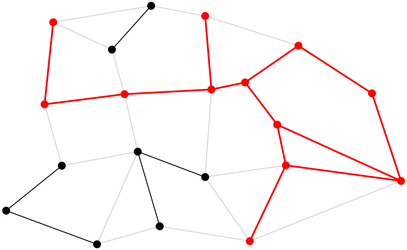

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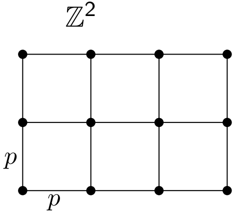

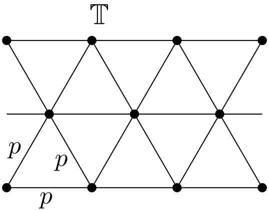

$p < p_c$, a.s. no infinite component;
$p > p_c$, a.s. existence of an infinite component.

Criticality: $p_c(\mathbb{Z}^2) = \frac{1}{2}$. $p_c(\mathbb{T}) = 2\sin\frac{\pi}{18}$.

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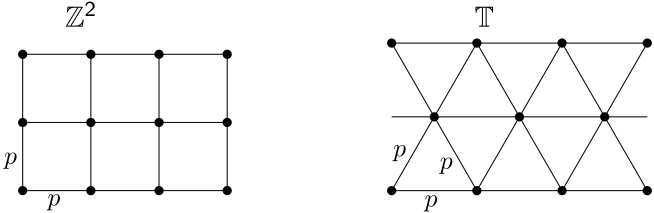

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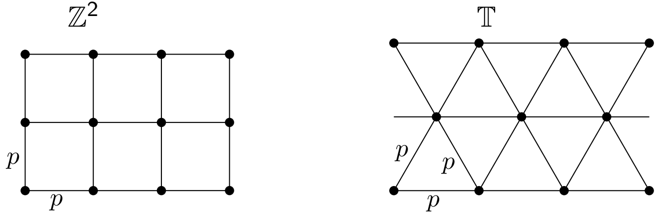

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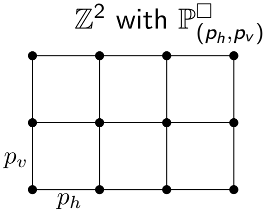
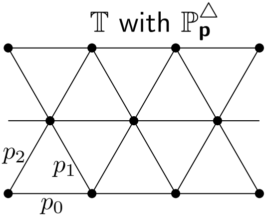
Inhomogeneous bond percolation





Criticality for $\mathbb{Z}^2$: $p_v + p_h = 1$.
Criticality for $\mathbb{T}$: $\kappa_\triangle(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1$,
$(\mathbf{p} = (p_0, p_1, p_2) \in [0,1)^3)$.

Call $\mathcal{M}$ the above class of critical (inhomogeneous) models.

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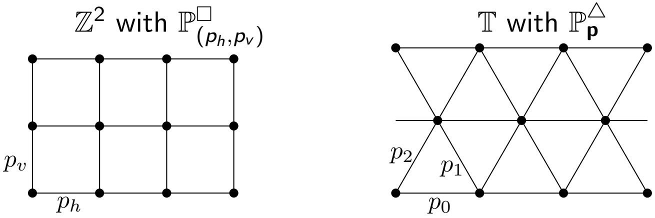

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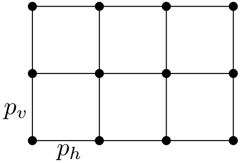
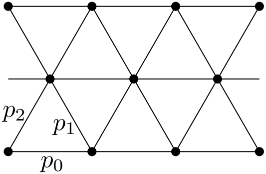
Inhomogeneous bond percolation

$\mathbb{Z}^2$ with $\mathbb{P}^{\square}_{(p_h, p_v)}$



$\mathbb{T}$ with $\mathbb{P}^{\triangle}_{\mathbf{p}}$



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Criticality

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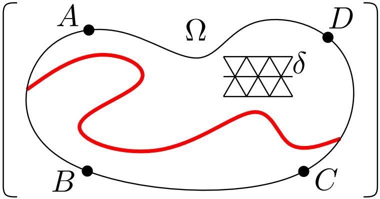

$\to D(\Omega, A, B, C, D)$, as $\delta \to 0$

where $D(\Omega, A, B, C, D)$ is conformally invariant and does not depend on the underlying model.

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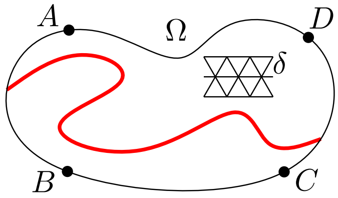

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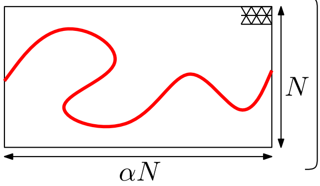
The box-crossing property

A model satisfies the box-crossing property if for all α there exists $c(\alpha) > 0$ s.t. for all N:

$$\mathbb{P}\left[\ \cdot\ \right] \in [c(\alpha), 1 - c(\alpha)]$$



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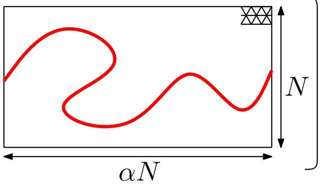
$\mathbb{P}$ [] $\in [c(\alpha), 1 - c(\alpha)]$

The homogeneous models in $\mathcal{M}$ satisfy the box-crossing property.

Main result I

Theorem

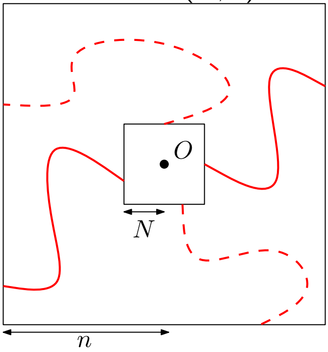
All models in $\mathcal{M}$ satisfy the box-crossing property.

Exponents at criticality

For a critical percolation measure $\mathbb{P}_{\mathbf{p}_c}$, as $n \to \infty$, we expect:

- volume exponent: $\mathbb{P}_{\mathbf{p}_c}(|C_0| = n) \approx n^{-1-1/\delta}$,
- connectivity exponent: $\mathbb{P}_{\mathbf{p}_c}(0 \leftrightarrow x) \approx |x|^{-\eta}$,
- one-arm exponent: $\mathbb{P}_{\mathbf{p}_c}(\mathrm{rad}(C_0) = n) \approx n^{-1-1/\rho}$,
- $2j$-alternating-arms exponents: $\mathbb{P}_{\mathbf{p}_c}[A_{2j}(N, n)] \approx n^{-\rho_{2j}}$,

$A_4(N, n)$



Exponents near ciritcality

- Percolation probability: $\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0| = \infty) \approx \epsilon^{\beta}$ as $\epsilon \downarrow 0$,
- Correlation length: $\xi(\mathbf{p}_c - \epsilon) \approx \epsilon^{-\nu}$ as $\epsilon \downarrow 0$,
  where $-\frac{1}{n} \log \mathbb{P}_{\mathbf{p}_c-\epsilon}(\mathrm{rad}(C_0) \geq n) \to_{n\to\infty} \frac{1}{\xi(\mathbf{p}_c-\epsilon)}$.
- Mean cluster-size: $\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0|; |C_0| < \infty) \approx |\epsilon|^{-\gamma}$ as $\epsilon \to 0$,
- Gap exponent: for $k \geq 1$, as $\epsilon \to 0$,

$$\frac{\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0|^{k+1}; |C_0| < \infty)}{\mathbb{P}_{\mathbf{p}_c+\epsilon}(|C_0|^{k}; |C_0| < \infty)} \approx |\epsilon|^{-\Delta}.$$

Scaling relations

- Kesten '87. For models with the box-crossing property if ρ or η exist, then
$$\eta\rho = 2 \quad \text{and} \quad 2\rho = \delta + 1.$$
- Kesten '87. For models with the box-crossing property rotation and translation invariance, β, ν, γ and δ may be expressed in terms of ρ and ρ_4.

Main result II

Theorem

If one of the arm exponents exists in one of the models in $\mathcal{M}$, then it exists and is the same in all models in $\mathcal{M}$.

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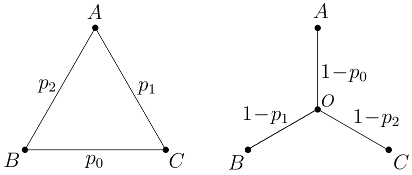
If ρ and ρ_4 exist in one of the models in $\mathcal{M}$, then the exponents away form criticality exist and are the same in the critical homogeneous models on the square, triangular and hexagonal lattices.

Star–triangle transformation



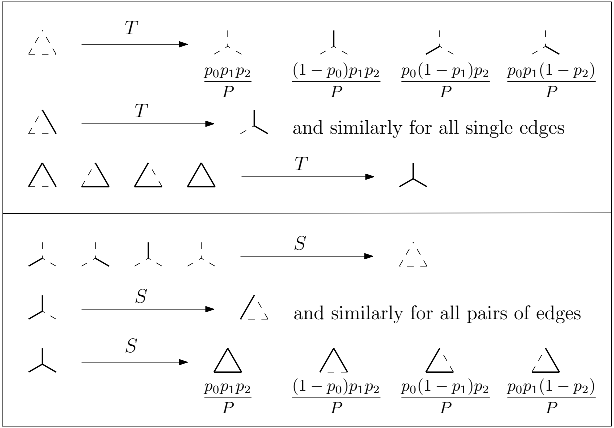
Take ω, respectively ω', according to the measure on the left, respectively right. The families of random variables

$$\left(x \xleftrightarrow{G,\omega} y : x, y = A, B, C\right), \quad \left(x \xleftrightarrow{G',\omega'} y : x, y = A, B, C\right),$$

have the same joint law whenever

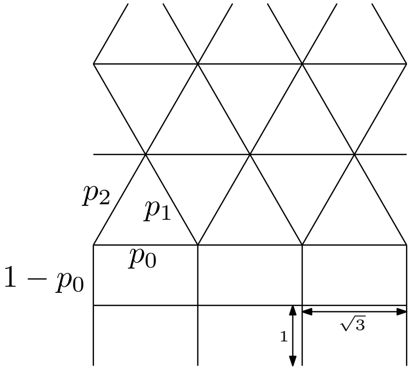
$$\kappa_\triangle(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1.$$

Coupling



where $P = (1-p_0)(1-p_1)(1-p_2)$.

Lattice transformation



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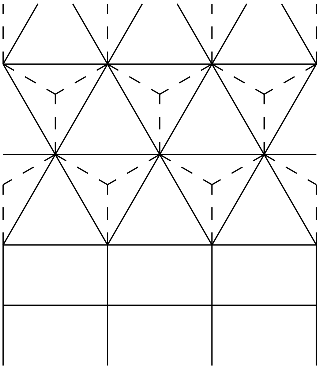

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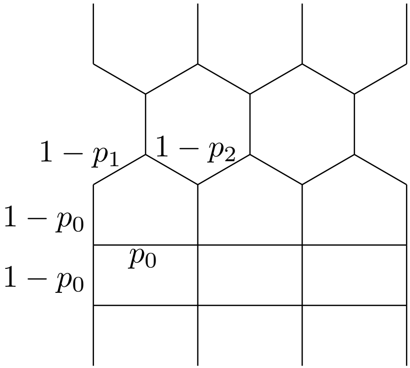

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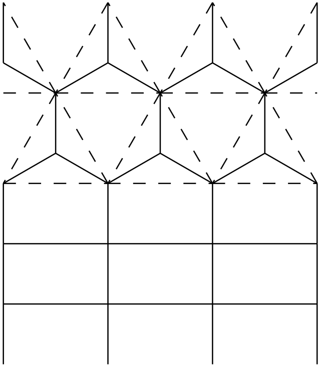

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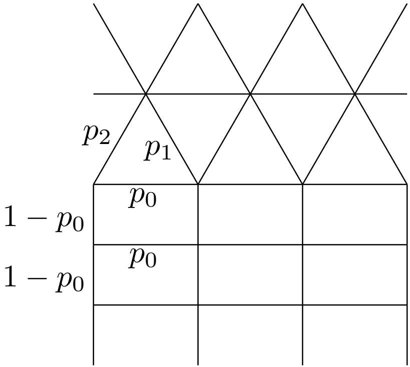

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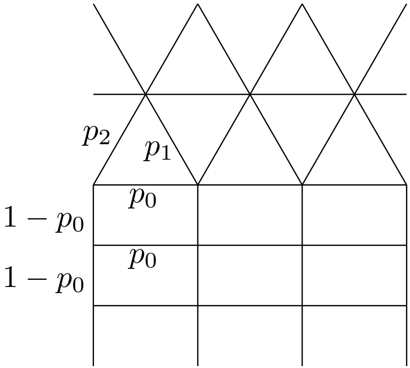
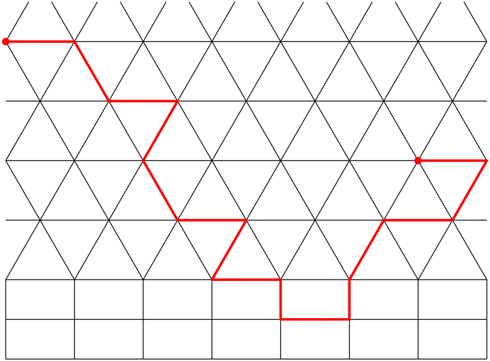


The measure is preserved.

Transformation of paths



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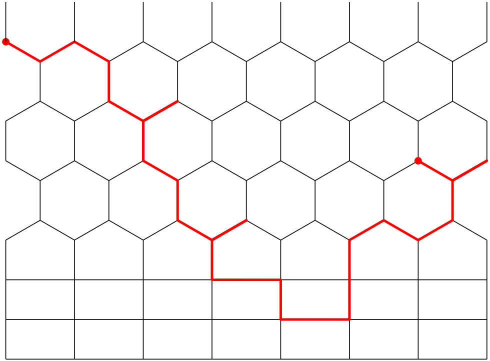

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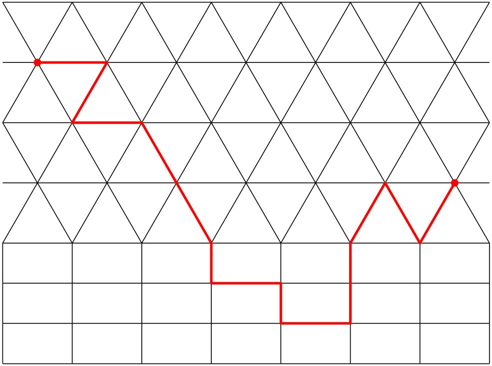

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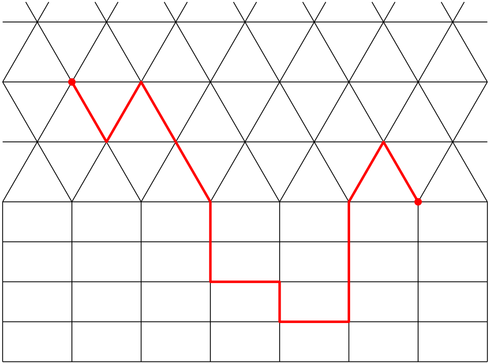

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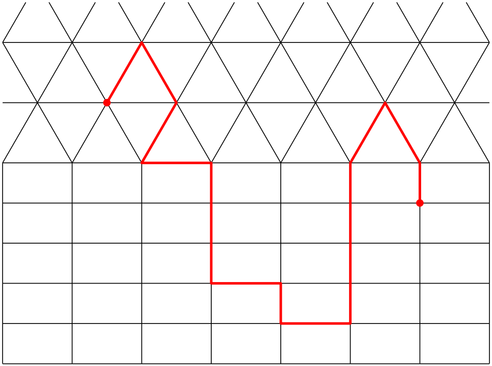

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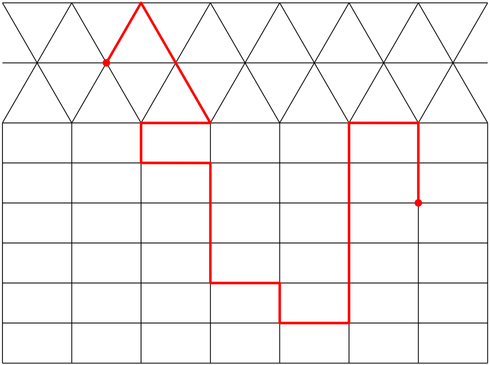

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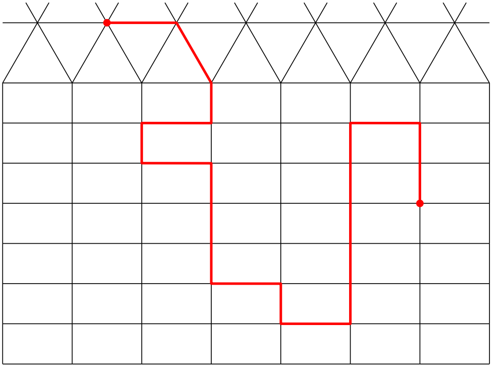

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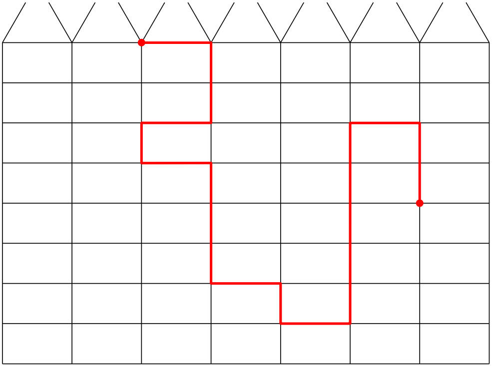

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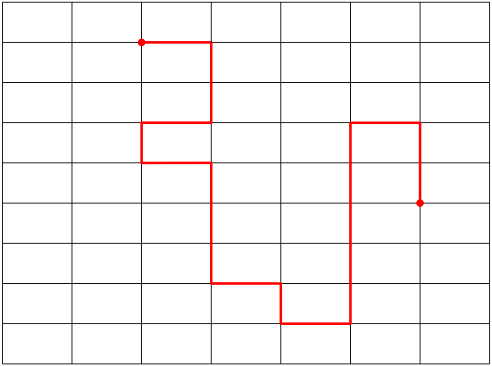

Transporting box crossings

Proposition

For $\mathbf{p} = (p_0, p_1, p_2) \in (0,1)^3$ *such that* $\kappa_\triangle(\mathbf{p}) = 1$,
$\mathbb{P}^\square_{(p_0, 1-p_0)}$ *satisfies the box-crossing property iff* $\mathbb{P}^\triangle_{\mathbf{p}}$ *does.*

Use of the proposition: $\mathbb{P}^\square_{\left(\frac{1}{2},\frac{1}{2}\right)}$ satisfies the box-crossing property,

hence so does $\mathbb{P}^\triangle_{\left(\frac{1}{2},p_0,p_0'\right)}$, when $\kappa_\triangle\left(\frac{1}{2}, p_0, p_0'\right) = 1$,

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Transporting arm exponents

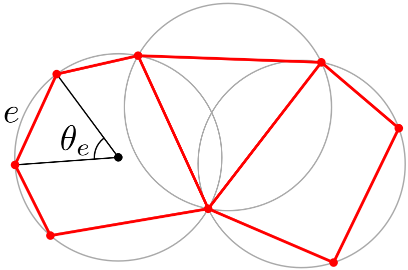
Proposition

For any $k \in \{1, 2, 4, 6, \dots\}$ *and any self-dual triplet* $\mathbf{p} \in [0,1)^3$ *with* $p_0 > 0$, *there exist* $c_0, c_1, n_0 > 0$ *such that, for all* $n \geq n_0$,

$$c_0 \mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)] \leq \mathbb{P}^{\square}_{(p_0, 1-p_0)}[A_k(n)] \leq c_1 \mathbb{P}^{\triangle}_{\mathbf{p}}[A_k(n)].$$

Using the same procedure we transport arm exponents between models.

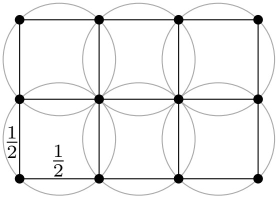
Isoradial graphs

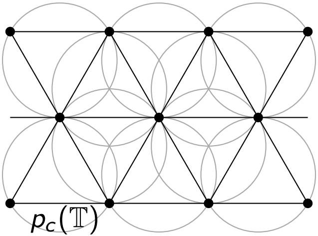


Each face is inscribed in a circle of radius 1.

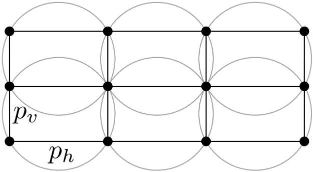
$$\frac{p_e}{1-p_e} = \frac{\sin\left(\frac{\pi-\theta(e)}{3}\right)}{\sin\left(\frac{\theta(e)}{3}\right)}.$$

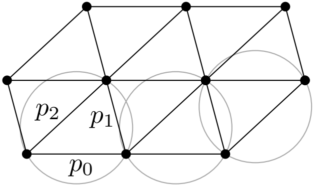
Inhomogeneous models as isoradial graphs





Inhomogeneous models as isoradial graphs





$$p_v + p_h = 1, \qquad \kappa_\triangle(\mathbf{p}) = p_0 + p_1 + p_2 - p_0 p_1 p_2 = 1$$

Conjectures

Conjecture

The class $\mathcal{M}$ may be extended to all isoradial graphs.

Conjectures

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The class $\mathcal{M}$ may be extended to all isoradial graphs.

Conjecture

The class $\mathcal{M}$ may be extended to periodic isoradial graphs.

Thank you!