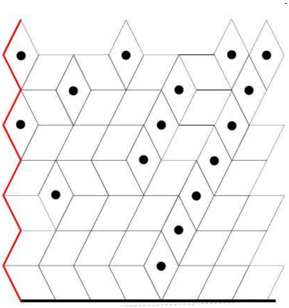
Interlacing Particle Systems and the Gaussian Free Field

Jeffrey Kuan

Harvard University

30 August 2011

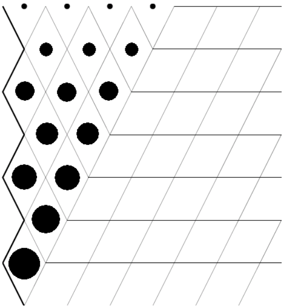
The particles live on a lattice in the quarter plane. There are $\left[\frac{j+1}{2}\right]$ particles on the jth level. The particles must satisfy an interlacing property.

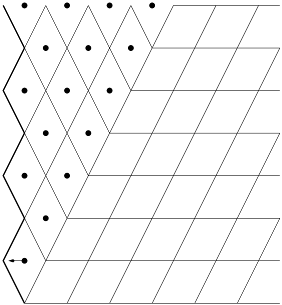


It looks 3D, so we can define a height function.

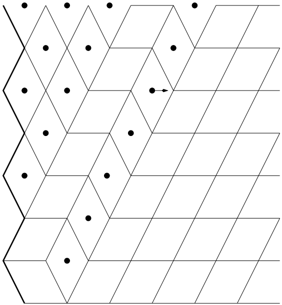
Define a Markov Chain as follows:

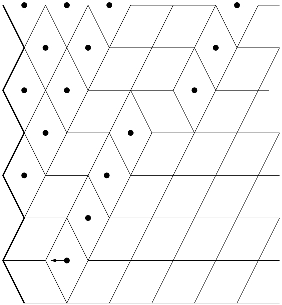
It starts at $t = 0$ with the densely packed configuration. Imagine that particles have weights that decrease upwards.

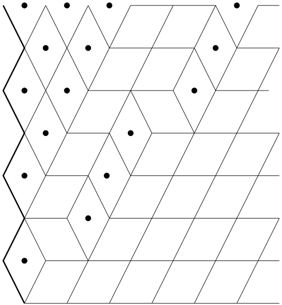




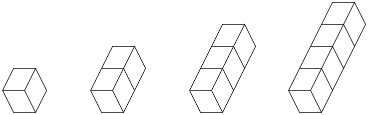
Each particle tries to jump to the left and to the right independently with rate 1/2, with the wall acting as a reflecting barrier. It is blocked by heavier particles and it can push lighter particles.







Motivations:
In terms of the stepped surface in 3d, this can be viewed as adding and removing sticks



This model falls into the Anisotropic Kardar-Parisi-Zhang universality class from mathematical physics.

Connections to representation theory of Lie groups. In particular, this system corresponds to representations of the orthogonal groups. Previous work has been done for the unitary groups (Borodin-K), as well as for the symplectic groups (Windridge). The discrete-time case also involves representation theory (Defosseux).

The sine kernel, Airy kernel and Pearcey kernel appear after appropriate rescaling.

An animation can be found at
`http://www.math.harvard.edu/~jkuan/Animation.html`

Given a domain $D \subset \mathbb{R}^d$, let $H_s(D)$ be the space of smooth, real-valued functions that are supported on a compact subset of D. Give $H_s(D)$ the Dirichlet inner product, and let $H(D)$ be its Hilbert space completion.

If $\{e_i\}$ is an orthonormal basis for $H(D)$ and $\{\alpha_i\}$ are i.i.d. $\mathcal{N}(0,1)$, then $h = \alpha_1 e_1 + \alpha_2 e_2 + \ldots$ diverges a.s. as an element of $H(D)$. However, for $f \in H(D)$, $\langle h, f\rangle_\nabla$ is a well-defined Gaussian with mean zero and variance $\langle f, f\rangle_\nabla$.

Using integration by parts, it is equivalent to define it as follows.

Definition

A **Gaussian free field** on D is a family of mean zero Gaussian random variables, indexed by $f \in (-\Delta)H(D)$, denoted by (h, f). Their covariance is

$$\mathbb{E}[(h,f)(h,g)] = \int_{D\times D} G(x,y)f(x)g(y)dxdy,$$

where G is the Green's function for the Laplacian on D with Dirichlet boundary conditions.

Example. For $t \in (0, \infty) = D$, let B_t denote (h, δ_t). The Green's function is $G(x, y) = x \wedge y$. Then

1. $B_0 = 0 \quad a.s.$
2. For $t > s$, $B_t - B_s$ is normal with mean zero and variance $t - s$.
3. $B_t - B_s$ and B_s are independent.

So the Gaussian free field on $(0, \infty)$ is Brownian motion. The Gaussian free field is considered to be a universal object the same way that Brownian motion is.

In higher dimensions, $G(x,x)$ is undefined, so the GFF at a point is undefined. However, for distinct $x_1,\ldots,x_k$ we can formally write

$$\mathbb{E}[\langle h,\delta_{x_1}\rangle\langle h,\delta_{x_2}\rangle] = G(x_1,x_2)$$

In general, if $X_1,\ldots,X_k$ are mean zero random variables such that all of their linear combinations are Gaussian, then

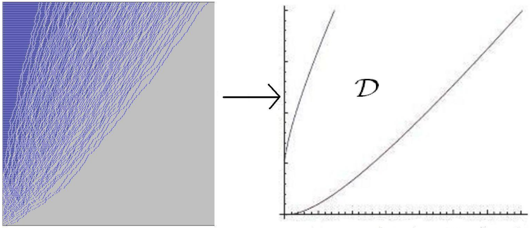
$$\mathbb{E}[X_1\ldots X_k] = \begin{cases} \sum_{\sigma}\prod_{j=1}^{k/2}\mathrm{Cov}(X_{\sigma(j)},X_{\sigma(j+1)}), & k \text{ even} \\ 0, \quad k \text{ odd}, \end{cases}$$

where the sum is over fixed-point-free involutions in S_k.

Therefore, we may write

$$\mathbb{E}[\langle h, \delta_{x_1} \rangle \cdots \langle h, \delta_{x_k} \rangle] = \begin{cases} \sum_{\sigma} \prod_{j=1}^{k/2} G(x_{\sigma(j)}, x_{\sigma(j+1)}), & k \text{ even} \\ 0, & k \text{ odd}, \end{cases}$$

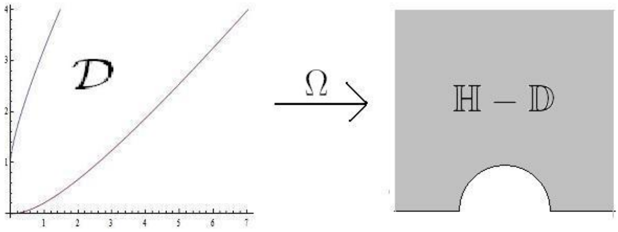
where the sum is over fixed-point-free involutions in S_k.



More precisely, $\mathcal{D} \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ is the set of all (ν, η, τ) such that

$$\lim_{L\to\infty} \mathbb{P}(\text{there is a particle at } ([\nu L], [\eta L]) \text{ at time } \tau L) \in (0, 1)$$

There is a map Ω from $\mathcal{D} \to \mathbb{H} - \mathbb{D}$.



Let $h(x, n, t)$ denote the height function at (x, n) at time t, and define $H_L : \mathcal{D} \to \mathbb{R}$ to be the fluctuations of the height function, i.e.
$H_L(\eta, \nu, \tau) = h([\eta L], [\nu L], \tau L) - \mathbb{E}[h([\eta L], [\nu L], \tau L)]$.

Theorem

For distinct $\varkappa_j = (\eta_j, \nu_j, \tau) \in \mathcal{D}$, *let* $\Omega_j = \Omega(\varkappa_j)$. *Then*

$$\mathbb{E}[H_L(\varkappa_1)\dots H_L(\varkappa_k)] = \begin{cases} \sum_\sigma \prod_{j=1}^{k/2} G(\Omega_{\sigma(j)}, \Omega_{\sigma(j+1)}), & k \text{ even} \\ 0, & k \text{ odd}, \end{cases}$$

where

$$G(z,w) = \frac{1}{2\pi} \log \frac{z + z^{-1} - \bar{w} - \bar{w}^{-1}}{z + z^{-1} - w - w^{-1}}$$

is the Green's function for the Laplacian on $\mathbb{H} - \mathbb{D}$ *with Dirichlet boundary conditions.*

The proof uses the fact that the particle system is a determinantal point process, and relies on the asymptotic expansion of the correlation kernel. In other words, for $(x_1, n_1) \dots (x_k, n_k)$,

$$\mathbb{P}(\text{there are particles at } (x_j, n_j) \text{ at time } t) = \det[K(x_i, n_i, x_j, n_j, t)]_1^k,$$

where K is called the correlation kernel. In a discrete setting, the probability measure is completely determined by K.

A more general statement is true.
Suppose

$$K(x_1, n_1, x_2, n_2, t) \approx \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} \int_{\Gamma_2} \frac{\exp(NS(\eta_1, \nu_1, \tau, u))}{\exp(NS(\eta_2, \nu_2, \tau, w))} f(u, w) dw du,$$

where Γ_1, Γ_2 are steepest descent paths.

Theorem

With technical assuumptions on S and f,

$$\mathbb{E}[H_L(\varkappa_1)\dots H_L(\varkappa_k)] \to \begin{cases} \sum_{\sigma} \prod_{j=1}^{k/2} G(\Omega_{\sigma(j)}, \Omega_{\sigma(j+1)}), & k \text{ even} \\ 0, & k \text{ odd}, \end{cases}$$

where

$$G(z,w) = \frac{1}{2\pi} \int_{\bar{z}}^{z} \int_{\bar{w}}^{w} \frac{f(u,v)f(v,u)}{S'_\nu(u)S'_\nu(v)} du dv,$$

with S'_ν denoting $(\partial^2/\partial\nu\partial z)S$.

In this specific case,

$$S(\nu,\eta,\tau;u)=\tau\frac{u+u^{-1}}{2}+\eta\log\left(\frac{u+u^{-1}}{2}-1\right)-\nu\log u,$$

$$f(u,v)=\frac{1}{v}\frac{1-u^{-2}}{v+v^{-1}-u-u^{-1}}$$

Conjectures:

- The single-point fluctuations of the height function should be logarithmic. (First predicted by D.E. Wolf for AKPZ, using renormalization group). In other words, there should be the convergence of moments:

$$\text{const}\,\frac{H_L(\kappa)}{\sqrt{\log L}} \to \mathcal{N}(0,1).$$

- We can define a pairing $\langle H_L, f\rangle$ so that the random vector $(\langle H_L, f_j\rangle)_{j=1}^k$ converges in distribution to Gaussian vector with mean zero and covariance matrix $\|(\nabla f_j, \nabla f_i)\|_1^k$.
- Extend to $\tau_1 \neq \tau_2$.