

The spectrum of dynamically defined operators

Helge Krüger

## Introduction

The main claim
Pictures
The unitary case

## The spectrum of dynamically defined operators

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## Schrödinger operators

The spectrum of dynamically defined operators

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Introduction
The main claim
Pictures
The unitary case
Anderson localization

The discrete Laplacian acting on the square summable sequences $\ell^{2}(\mathbb{Z})$ is given by

$$
\begin{equation*}
\Delta \psi(n)=\psi(n+1)+\psi(n-1) . \tag{1}
\end{equation*}
$$

For a potential, i.e. a bounded sequence, $V: \mathbb{Z} \rightarrow \mathbb{R}$, we call $H=\Delta+V$ a Schrödinger operator.


For $V(n)=2 \lambda \cos (2 \pi(n \omega+x))$ with $\omega$ irrational and $\lambda \neq 0$, we have the Almost-Mathieu operator. Then the spectrum is alwavs a Cantor set.

## Schrödinger operators

The spectrum of dynamically defined operators

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For $V(n)$ i.i.d.r.v. with distribution supported in $[a, b]$, $H=\Delta+V$ is called the Anderson model. we have that the spectrum is given by

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\begin{equation*}
\sigma(H)=\operatorname{range}(V)+\sigma(\Delta)=[a-2, b+2] . \tag{2}
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The spectrum of dynamically defined operators

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Number theory of
$\omega n(\bmod 1)$ and $\omega n^{2}(\bmod 1)$

The spectrum of dynamically defined operators
$\omega n(\bmod 1)$ and $\omega n^{2}(\bmod 1)$ are both equidistributed in $[0,1]$
Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization

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The spectrum of dynamically defined operators

Helge Krüger

Introduction

## The main claim

Pictures
The unitary case
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\left\{\beta_{1}<\beta_{2}<\cdots<\beta_{N}\right\}=\{\omega n \quad(\bmod 1)\}_{n=1}^{N}
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and

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\left\{\gamma_{1}<\gamma_{2}<\cdots<\gamma_{N}\right\}=\left\{\omega n^{2} \quad(\bmod 1)\right\}_{n=1}^{N}
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## Number theory of

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The spectrum of dynamically defined operators

Helge Krüger

Introduction

The main claim
Pictures
Thie unitary case
Andersen localization
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Then the set of lengths

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\left\{I_{j}=\beta_{j+1}-\beta_{j}, \quad j=1, \ldots, N-1\right\}
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The spectrum of dynamically defined operators

Helge Krüger

Introduction

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obey Poisson statistics for generic $\omega$.

## The skew-shift Schrödinger operator

The spectrum of
dynamically defined operators

Helge Krüger

Introduction
The main claim Pictures

The unitary case

Define the skew-shift $T: \mathbb{T}^{2} \rightarrow \mathbb{T}^{2}, \mathbb{T}=\mathbb{R} / \mathbb{Z}$,

$$
\begin{equation*}
T(x, y)=(x+2 \omega, x+y)(\bmod 1) . \tag{3}
\end{equation*}
$$

Then $\omega n^{2}=T^{n}(\omega, 0)_{2}(\bmod 1)$.
It thus makes sense instead of considering the potential $V(n)=f\left(n^{2} \omega(\bmod 1)\right)$, to consider potentials given by
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Conjecture: For sufficiently regular $f$
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The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim Pictures

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\begin{equation*}
V(n)=\lambda f\left(T^{n}(x, y)\right) \tag{4}
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for $f: \mathbb{T}^{2} \rightarrow \mathbb{R}$. These then form an ergodic family of potentials.


## The skew-shift Schrödinger operator

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim Pictures The unitary case Anderson focalization Fiture work

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Progress:
Bourgain,
Bourgain-Jitomirskaya, K


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Conjecture: For sufficiently regular $f$, the spectrum of $\Delta+V$ consists of finitely many intervals and is Anderson localized. This means it behaves as in the random case.

Progress: Large coupling $(\lambda \gg 1)$ : Bourgain-Goldstein-Schlag, Bourgain, Bourgain-Jitomirskaya, K. Small coupling ( $0<\lambda \ll 1$ ): Bourgain. largely open Necessity of regularity: Avila-Bochi-Damanik, Boshernitzan-Damanik.


The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization Future work

$$
V(n)=2 \lambda \cos \left(2 \pi \omega n^{2}\right)
$$

$$
\begin{aligned}
& \lambda=0.9 \\
& \omega=\sqrt{2}
\end{aligned}
$$

$$
H: \ell^{2}([1,50]) \rightarrow \ell^{2}([1,50])
$$

$$
\left(\begin{array}{cccc}
b(1) & 1 & & \\
1 & b(2) & 1 & \\
& \ddots & \ddots & \ddots \\
& & 1 & b(50)
\end{array}\right)
$$

$$
H u_{j}=E_{j} u_{j} \text { for } j=1, \ldots, 50
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The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization
Future work

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Black dot at $\left(n, E_{j}\right)$ if $\left|u_{j}(n)\right| \geq 0.01$.



The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson tocalization
Fiture work

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Energy


## Verblunsky coefficients and orthogonal polynomials on the unit circle

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case

It turns out that this problem can be solved explicitely for the unitary analog of Schrödinger operators: CMV matrices.

Define a sequence of monic polynomials by the Szegő recursion

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The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case

It turns out that this problem can be solved explicitely for the unitary analog of Schrödinger operators: CMV matrices.

Given a sequence of Verblunsky coefficients

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\begin{equation*}
\alpha(n) \in \mathbb{D}=\{z \in \mathbb{C}: \quad|z|<1\} . \tag{5}
\end{equation*}
$$

Define a sequence of monic polynomials by the Szegő recursion

$$
\begin{equation*}
\Phi_{n+1}(z)=z \Phi_{n}(z)-\overline{\alpha(n)} \Phi_{n}^{*}(z) \tag{6}
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$$

with $\Phi_{n}^{*}(z)=z^{n} \overline{\Phi_{n}\left(\bar{z}^{-1}\right)}$.

## Verblunsky coefficients and orthogonal polynomials on the unit circle

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures

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Then there exists an unique probability measure $\mu$ supported on $\partial \mathbb{D}$ such that $\Phi_{n}$ are the polynomials obtained by orthogonalizing $1, z, \ldots$ in $L^{2}(\partial \mathbb{D}, \mu)$.

## Verblunsky coefficients and orthogonal polynomials on the unit circle

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization Future work

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Then there exists an unique probability measure $\mu$ supported on $\partial \mathbb{D}$ such that $\Phi_{n}$ are the polynomials obtained by orthogonalizing $1, z, \ldots$ in $L^{2}(\partial \mathbb{D}, \mu)$.
One can also view $\mu$ as the spectral measure of the CMV matrix $\mathcal{C}$ corresponding to the Verblunsky coefficients $\alpha(n)$.

## Rotating the Verblunsky coefficients

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization

Recall

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\Phi_{n+1}(z)=z \Phi_{n}(z)-\overline{\alpha(n)} \Phi_{n}^{*}(z) . \tag{7}
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$$

Define $\tilde{\alpha}(n)=\mathrm{e}^{-2 \pi \mathrm{i} \theta(n+1)} \alpha(n)$. Then one has for $\widetilde{\Phi}_{n}(z)=\mathrm{e}^{2 \pi \mathrm{i} n \theta} \Phi_{n}\left(\mathrm{e}^{-2 \pi \mathrm{i} \theta} z\right)$ that

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Hence we see that the measure $\tilde{\mu}$ corresponding to $\tilde{\alpha}$ is just the measure $\mu$ rotated by $\mathrm{e}^{2 \pi \mathrm{i} \theta}$.

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The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization

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In particular, that

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\sigma(\mathcal{C})=\mathrm{e}^{2 \pi \mathrm{i} \theta} \sigma(\widetilde{\mathcal{C}}) . \tag{9}
\end{equation*}
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## Skew-shift Verblunsky coefficients

The spectrum of dynamically defined operators

Helge Krüger

Given $\lambda \in \mathbb{D}$, define the function $f(x, y)=\lambda \mathrm{e}^{2 \pi \mathrm{i} y}$ and the Verblunsky coefficients

$$
\begin{equation*}
\alpha_{x, y}(n)=f\left(T^{n}(x, y)\right)=\lambda \mathrm{e}^{2 \pi \mathrm{i}(\omega n(n-1)+n x+y)} \tag{10}
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The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case Anderson localization

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From the previous results, we have

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\begin{equation*}
\sigma\left(\mathcal{C}_{\tilde{x}, y}\right)=\mathrm{e}^{2 \pi \mathrm{i}(x-\tilde{x})} \sigma\left(\mathcal{C}_{x, y}\right) \tag{11}
\end{equation*}
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From minimality of the skew-shift, we have

Since $\sigma\left(\mathcal{C}_{x, y}\right) \subseteq \partial \mathbb{D}$ is non-empty, we obtain

## Theorem

For every $x, y$, we have

## Skew-shift Verblunsky coefficients

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case

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The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim Pictures

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## Anderson localization

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The-main claim
Pictures
The unitary case
Anderson localization

Using similar arguments, one also obtains

## Theorem

Let $\lambda \in \mathbb{D}$ and $x \in \mathbb{R}$. For almost-every $y \in \mathbb{R}$, the CMV matrix $\mathcal{C}_{x, y}$ has pure point spectrum with exponentially decaying eigenfunctions.

This is what is called Anderson localization.
Proof:

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## Anderson localization

The spectrum of dynamically defined operators

Helge Krüger

Introduction
The main claim
Pictures
The unitary case
Anderson localization

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Proof: Define the Lyapunov exponent

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\begin{equation*}
L(z)=\lim _{N \rightarrow \infty} \frac{1}{N} \int_{\mathbb{T}^{2}} \log \left\|A_{N}^{z}(x, y)\right\| d(x, y) \tag{14}
\end{equation*}
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where $A_{N}^{z}(x, y)=A^{z}\left(T^{n}(x, y)\right) \cdots A^{z}(x, y)$ is the transfer matrix

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A^{z}(x, y)=\frac{1}{\sqrt{1-|\lambda|^{2}}}\left(\begin{array}{cc}
z & -\bar{\lambda} \mathrm{e}^{-2 \pi \mathrm{i} y}  \tag{15}\\
-\lambda \mathrm{e}^{2 \pi \mathrm{i} y} z & 1
\end{array}\right) .
$$

One then shows $L\left(e^{2 \pi i t}\right)=L\left(e^{2 \pi i s}\right)$ as before. Since $\alpha(n) \neq 0$ one thus must have $L\left(\mathrm{e}^{2 \pi i t}\right)>0$.

## Anderson localization

The spectrum of dynamically defined operators

Helge Krüger

Introduction The main claim Pictures The unitary case

Anderson localization

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The spectrum of dynamically defined operators

Helge Krüger

Introduction The main claim Pictures The unitary case
Anderson localization Future work

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The spectrum of dynamically defined operators

Helge Krüger

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\end{equation*}
$$

where $A_{N}^{z}(x, y)=A^{z}\left(T^{n}(x, y)\right) \cdots A^{z}(x, y)$ is the transfer matrix

$$
A^{z}(x, y)=\frac{1}{\sqrt{1-|\lambda|^{2}}}\left(\begin{array}{cc}
z & -\bar{\lambda} \mathrm{e}^{-2 \pi \mathrm{i} y}  \tag{15}\\
-\lambda \mathrm{e}^{2 \pi \mathrm{i} y} z & 1
\end{array}\right)
$$

One then shows $L\left(\mathrm{e}^{2 \pi \mathrm{i} t}\right)=L\left(\mathrm{e}^{2 \pi \mathrm{i} s}\right)$ as before. Since $\alpha(n) \neq 0$, one thus must have $L\left(\mathrm{e}^{2 \pi \mathrm{i} t}\right)>0$. Standard results then imply the localization claim.

## Future work

The spectrum of dynamically defined operators

Helge Krüger

It should be tedious but possible to extend these results to Verblunsky coefficients of the form

$$
\begin{equation*}
\alpha_{x, y}(n)=\lambda \mathrm{e}^{2 \pi \mathrm{i}(\omega n(n-1)+n x+y)}+\varepsilon g\left(T^{n}(x, y)\right) \tag{16}
\end{equation*}
$$

where $g: \mathbb{T}^{2} \rightarrow \mathbb{R}$ is real-analytic and $\varepsilon>0$ is small enough.

More interestingly, one should be able to compute the eigenvalue statistics of this operator

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