Construction of a Massless Quantum Field Theory over the p-adics

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- Cannot talk about "$\phi(x)$", no coordinate process. For any $f \in \mathcal{S}$ we have a random variable:

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- (Gaussian) Examples: White Noise, Continuum GFF with $d > 2$

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- (d) κ scale invariant: "$|\lambda|^{-\kappa}\phi(\frac{x}{\lambda}) \overset{d}{=} \phi(x)$" for all $\lambda \in \mathbb{K}^*$.
- (e) Non-trivial (Non-Gaussian): $\left(\frac{d}{dz}\right)^4\Big|_{z=0} \log\left(\mathbb{E}\left[e^{z\phi(f)}\right]\right) \neq 0$ for some $f \in \mathcal{S}$.

Theorem (Abdesselam, C., Guadagni): Let p be prime and set $\mathbb{K} = \mathbb{Q}_p$, $d = 3$. There exists an $L_0 > 1$ such that for any $L = p^j > L_0$ there exists a measure ν (on $\mathcal{S}'(\mathbb{Q}_p^3)$) satisfying (a)-(c) and (e) along with a restricted version of (d) (ν satisfies scale invariance with some κ for any λ of the form L^k, $k \in \mathbb{Z}$).

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Earlier Work:

(Bleher, Sinai 1973), (Collet, Eckmann 1977): Hierarchical model, non-trivial fixed point

(Gawedzki, Kupianien 1983 and 1984): Hierarchical model, non-trivial fixed point and nonzero connected four point function

(Brydges, Mitter, Scoppola 2003): Euclidean model, non-trivial fixed point

(Abdesselam 2006): Euclidean model, construction of a trajectory between Gaussian and non-trivial fixed points

Constructing a singular pertubation of the Gaussian μ_C with the covariance $C : \mathbb{K}^3 \times \mathbb{K}^3 \to \mathbb{R}$:

$$C(x,y) = C(x-y) = \frac{A}{|x-y|^{\frac{3-\epsilon}{2}}}$$
$$= \frac{A}{|x-y|^{2[\phi]}} \quad [\phi] = \frac{(3-\epsilon)}{4} = \kappa$$

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Formally the pertubation is given by:

$$\exp\left[-g \int_{\mathbb{K}^3} d^3x \; \phi^4(x)\right]$$

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$$\text{Regularize!: }\begin{cases}\text{Replace } C \text{ with } C_r \text{ where } \hat{C}_r(k) = \mathbb{1}\{|k| \le L^{-r}\}\hat{C}(k)\\ \text{Put the interaction in a finite box } \Lambda_s \text{ where } vol(\Lambda_s) = L^{3s}\end{cases}$$

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Define the following sequence of cutoff measures for $-\infty < r < s < \infty$:

$$d\nu_{r,s}(\tilde{\phi}) = \frac{1}{\mathcal{Z}_{r,s}} \exp\left[-\int_{\Lambda_s} d^3x\ \tilde{g}_r : \tilde{\phi}^4(x) :_{C_r} + \tilde{\mu}_r : \tilde{\phi}^2(x) :_{C_r}\right] d\mu_{C_r}(\tilde{\phi})$$

We'll construct our measure via the following limit:

$$\lim_{r \to -\infty} \lim_{s \to \infty} \nu_{r,s}$$

RG and its associated dynamical system

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If we are trying to calculate the partition function we see:

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\mathcal{Z}_{r,s} &= \int_{\mathcal{S}'} \exp\left[-\int_{\Lambda_s} d^3x\ \tilde{g}_r : \tilde{\phi}^4(x) :_{C_r} + \tilde{\mu}_r : \tilde{\phi}^2(x) :_{C_r}\right] d\mu_{C_r}(\tilde{\phi}) \\
&= \int_{\mathcal{S}'} \exp\left[-\int_{\Lambda_{s-r}} d^3x\ L^{(3-4[\phi])r}\tilde{g}_r : \phi^4(x) :_{C_0} + L^{(3-2[\phi])r}\tilde{\mu}_r : \phi^2(x) :_{C_0}\right] d\mu_{C_0}(\phi)
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&= \int_{\mathbb{R}^{Lat(\Lambda_{s-r})}} \prod_{x\in Lat(\Lambda_{s-r})} F_0\left(\phi(x)\right) d\mu_{C_0}(\phi)
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Integrating short range fluctuations

We define the fluctuation covariance Γ as follows:

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We integrate out Γ and rescale to define a sequence of effective actions. Given a function $F_j : \mathbb{R} \to \mathbb{R}$ we define F_{j+1} and δb_{j+1} in a way so that the following equation is satisfied:

$$\int \prod_{x \in Lat(\Lambda_{s-r-j})} F_j(\phi(x)) d\mu_{C_0}(\phi) = \int \prod_{x \in Lat(\Lambda_{s-r-(j+1)})} \left(F_{j+1}(\phi(x)) e^{\delta b_{j+1}} \right) d\mu_{C_0}(\phi)$$

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This gives us the definition:

$$\text{For } F_{j+1}(\phi(0)) = \int d\mu_\Gamma(\zeta) \prod_{y \in Lat(\Lambda_1)} F_j(L^{-[\phi]}\phi(0) + \zeta(y)) e^{-\delta b_{j+1}}$$

Preserved Functional Form

$$F_j(\phi) = \exp\left[-g_j : \phi^4(x) :_{C_0} - \mu_j : \phi^2 :_{C_0}\right] + K_j(\phi) \leftrightarrow (g_j, \mu_j, K_j) \in \mathbb{R} \times \mathbb{R} \times \mathcal{C}^9$$

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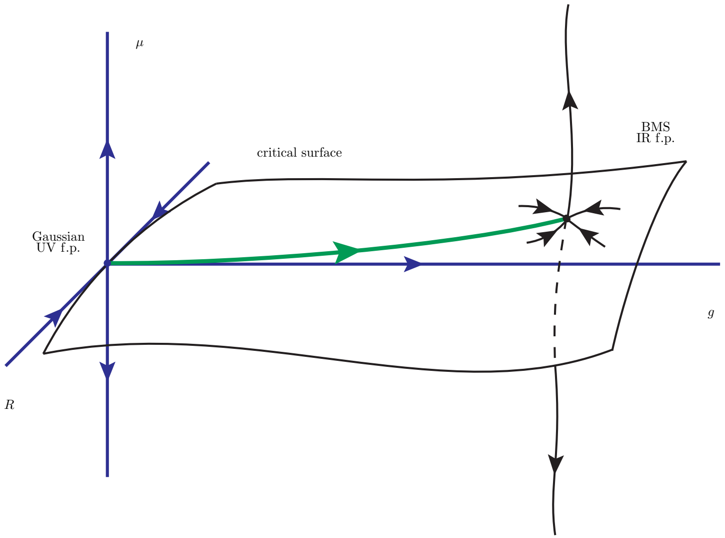
$$g_{j+1} = L^{\epsilon} g_j - a(L) L^{2\epsilon} g_j^2 + \cdots$$

$$\mu_{j+1} = L^{\frac{(3+\epsilon)}{2}} \mu_j + \cdots$$

$$\|K_{j+1}\| \leq L^{-\frac{1}{4}} \|K_j\|$$

Sketch of the RG phase portrait:



Existence of the Critical Mass

Analog of BMS Fixed point (g_*, μ_*, K_*), $g_* > 0$
Stable Manifold: $h(g, K) = \mu_{critical}$
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We now know how to choose $\{\tilde{g}_r, \tilde{\mu}_r\}_{-\infty \leq r \leq 0}$:
Pick some g_0 near g_*
Choose $\tilde{g}_r$ and $\tilde{\mu}_r$ so that:

$$g_0 = L^{(3-4[\phi])r} \tilde{g}_r, \ \ h(g_0, 0) = L^{(3-2[\phi])r} \tilde{\mu}_r$$

This puts F_0 on the stable manifold

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$$\log\left(\mathcal{Z}_{r,s}\right) = \sum_{j=1}^{(s-r-1)} vol(\Lambda_{r-s-j})\delta b_j + \int d\mu_{C_0}(\phi) F_{s-r}(\phi(0))$$

Can easily see existence of pressure for fixed UV cut-off

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To construct the measure we introduce a source term:

$$\mathcal{Z}_{r,s}(\tilde{f}) = \int_{\mathcal{S}'} \exp\left[-\int_{\Lambda_s} d^3x\ \tilde{g}_r : \tilde{\phi}^4(x) :_{C_r} + \tilde{\mu}_r : \tilde{\phi}^2(x) :_{C_r} + \tilde{f}(x)\tilde{\phi}(x)\right] d\mu_{C_r}(\tilde{\phi})$$

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$$\text{"}\ \mathbb{E}\left[e^{\phi(\tilde{f})} \right]\ \text{"} = \lim_{r\to-\infty} \lim_{s\to\infty} \frac{\mathcal{Z}_{r,s}(\tilde{f})}{\mathcal{Z}_{r,s}(0)}$$