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GÖDEL AND SET THEORY

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Kurt Gödel (1906–1978) with his work on the constructible universe L established the relative consistency of the Axiom of Choice (AC) and the Continuum Hypothesis (CH). More broadly, he ensured the ascendancy of first-order logic as the framework and a matter of method for set theory and secured the cumulative hierarchy view of the universe of sets. Gödel thereby transformed set theory and launched it with structured subject matter and specific methods of proof. In later years Gödel worked on a variety of set-theoretic constructions and speculated about how problems might be settled with new axioms. We here chronicle this development from the point of view of the evolution of set theory as a field of mathematics. Much has been written, of course, about Gödel's work in set theory, from textbook expositions to the introductory notes to his collected papers. The present account presents an integrated view of the historical and mathematical development as supported by his recently published lectures and correspondence. Beyond the surface of things we delve deeper into the mathematics. What emerges are the roots and anticipations in work of Russell and Hilbert, and most prominently the sustained motif of truth as formalizable in the "next higher system". We especially work at bringing out how transforming Gödel's work was for set theory. It is difficult now to see what conceptual and technical distance Gödel had to cover and how dramatic his re-orientation of set theory was. What he brought into set theory may nowadays seem easily explicated, but only because we have assimilated his work as integral to the subject. Much has also been written about Gödel's philosophical views about sets and his wider philosophical outlook, and while these may have larger significance, we keep the focus on the motivations and development of Gödel's actual mathematical constructions and contributions to set theory. Leaving his "concept of set" alone, we draw out how in fact he had strong mathematical instincts and initiatives, especially as seen in his last, 1970 attempt at the continuum problem.

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§1. From truth to set theory. Gödel's advances in set theory can be seen as part of a steady intellectual development from his fundamental work on completeness and incompleteness. Two remarkably prescient passages in his early publications serve as our point of departure. His incompleteness paper [1931], submitted for publication 17 November 1930, had a footnote 48a:

> As will be shown in Part II of this paper, the true reason for the incompleteness inherent in all formal systems of mathematics is that the formation of ever higher types can be continued into the transfinite (cf. D. Hilbert, "Über das Unendliche", Math. Ann. 95, p. 184), while in any formal system at most denumerably many of them are available. For it can be shown that the undecidable propositions constructed here become decidable whenever appropriate higher types are added (for example, the type ω to the system P). An analogous situation prevails for the axiom system of set theory.

This passage has been made much of,[1] whereas the following has not. It appeared in a summary [1932], dated 22 January 1931, of a talk on the incompleteness results given in Karl Menger's colloquium. Notably, matters in a footnote, perhaps an afterthought then, have now been expanded to take up fully one-third of an abstract on incompleteness:

> If we imagine that the system Z is successively enlarged by the introduction of variables for classes of numbers, classes of classes of numbers, and so forth, together with the corresponding comprehension axioms, we obtain a sequence (continuable into the transfinite) of formal systems that satisfy the assumptions mentioned above, and it turns out that the consistency (ω-consistency) of any of those systems is provable in all subsequent systems. Also, the undecidable propositions constructed for the proof of Theorem 1 [the Gödelian sentences] become decidable by the adjunction of higher types and the corresponding axioms; however, in the higher systems we can construct other undecidable propositions by the same procedure, and so forth. To be sure, all the propositions thus constructed are expressible in Z (hence are number-theoretic propositions); they are, however, not decidable in Z, but only in higher systems, for example, in that of analysis. In case we adopt a type-free construction of mathematics, as is done in the axiom system of set theory, axioms of cardinality (that is, axiom postulating the existence of sets of ever higher cardinality) take the place of type extensions, and it follows that certain arithmetic propositions that are undecidable in Z become decidable by axioms of

[1] See e.g., Kreisel [1980, pp. 183, 195, 197], a memoir on Gödel, and Feferman [1987], where the view advanced in the footnote is referred to as "Gödel's doctrine".

cardinality, for example, by the axiom that there exist sets whose cardinality is greater than every α_n, where $\alpha_0 = \aleph_0$, $\alpha_{n+1} = 2^{\alpha_n}$.

The salient points of these passages is that the addition of the next "higher type" to a formal system leads to newly provable propositions of the system; the iterative addition of higher types can be continued into the transfinite; and in set theory, new propositions become analogously provable from "axioms of cardinality". The transfinite heritage from Hilbert [1926], cited in footnote 48a, will be discussed in §5. Here we discuss the connections with the frameworks of types and of truth, which can be associated respectively with Bertrand Russell and Alfred Tarski.

Mathematical logic was emerging from the Russellian world of orders and types, and Gödel's work would reflect and transform Russell's initiatives. Russell's *ramified theory of types* is a scheme of logical definitions based on *orders* and *types* indexed by the natural numbers. Russell proceeded "intensionally"; he conceived this scheme as a classification of propositions based on the notion of *propositional function*, a notion not reducible to membership (extensionality). Proceeding however in modern fashion, we may say that the universe is to consist of *objects* stratified into disjoint types T_n, where T_0 consists of the *individuals*, $T_{n+1} \subseteq \{X \mid X \subseteq T_n\}$, and the types T_n for $n > 0$ are further ramified into orders O^i_n with $T_n = \bigcup_i O^i_n$. An object in O^i_n is to be defined either in terms of individuals or of objects in some fixed O^j_m for some $j < i$ and $m \leq n$, the definitions allowing for quantification only over O^j_m. This precludes Russell's Paradox and other "vicious circles", as objects can only consist of previous objects and are built up through definitions only referring to previous stages. However, in this system it is impossible to quantify over all objects in a type T_n, and this makes the formulation of numerous mathematical propositions at best cumbersome and at worst impossible. So Russell was led to introduce his axiom of *Reducibility*, which asserts that *for each object there is a predicative object having exactly the same constituents*, where an object is *predicative* if its order is the least greater than that of its constituents. This axiom in effect reduced consideration to individuals, predicative objects consisting of individuals, predicative objects consisting of predicative objects consisting of individuals, and so on—the *simple theory of types*.[2]

The above quoted Gödel passages can be considered a point of transition from type theory to set theory. The system P of footnote 48a is Gödel's streamlined version of Russell's theory of types built on the natural numbers as individuals, the system used in [1931]. The last sentence of the footnote calls to mind the other reference to set theory in that paper; Kurt Gödel [1931, p. 178] wrote of his comprehension axiom IV, foreshadowing his approach to

[2]In substantial criticism based on how mathematics ought to be regarded as a "calculus of extensions", Frank Ramsey [1926] emphasized and advocated this reduction.

set theory, "This axiom plays the role of [Russell's] axiom of reducibility (the comprehension axiom of set theory)." The system Z of the quoted [1932] passage is already the more modern first-order Peano arithmetic, the system in which Gödel in his abstract described his incompleteness results. The passage envisages the introduction of higher-type variables, which would have the effect of re-establishing the system P, but as one proceeds to higher and higher types, that "all the [unprovable] propositions constructed are expressible in Z (hence are number-theoretic propositions)" is an important point about incompleteness. The last sentence of the [1932] passage is Gödel's first remark on set theory of substance, and significantly, his example of an "axiom of cardinality" to take the place of type extensions is essentially the one that both Abraham Fraenkel [1922] and Thoralf Skolem [1923] had pointed out as unprovable in Ernst Zermelo's [1908] axiomatization of set theory and used by them to motivate the axiom of Replacement.

We next face head on the most significant underlying theme broached in our two quoted passages. Gödel's engagement with truth at this time, whether with conviction or caution,[3] could be viewed as his *entrée* into full-blown set theory. In later, specific terms, first-order satisfaction involves canvassing *arbitrary* variable assignments, and higher-order satisfaction requires, in effect, scanning all *arbitrary* subsets of a domain.

In the introduction to his dissertation on completeness Gödel [1929] had already made informal remarks about satisfaction, discussing the meaning of " 'A system of relations *satisfies* [*erfüllt*] a logical expression' (that is, the sentence obtained through substitution is true [*wahr*])." In a letter to Paul Bernays of 2 April 1931 Gödel[4] described how to define the unary predicate that picks out the Gödel numbers of the "correct" ["*richtig*"] sentences of first-order arithmetic. Gödel then remarked, as he would in similar vein several times in his career, "Simultaneously and independently of me (as I gathered from a conversation), Mr. Tarski developed the idea of defining the concept 'true proposition' in this way (for other purposes, to be sure)." Finally, Gödel emphasized the "decidability of the undecidable propositions in higher systems" specifically through the use of the truth predicate.

The semantic, recursive definition of the satisfaction relation, both first-order and higher-order, was first systematically formulated in set-theoretic terms by Tarski [1933][1935], to whom is usually attributed the undefinability of truth for a formal language within the language.[5] However, evident in Gödel's thinking was the *necessity* of a higher system to capture truth, and in fact Gödel maintained to Hao Wang [1996, p. 82] that he had come to the

[3]Cf. Feferman [1984].

[4]See Feferman and Dawson, Jr. [2003a, pp. 97ff].

[5]See Tarski [1933][1935] §5, theorem I; in a footnote Tarski wrote: "We owe the method used here to Gödel, who has employed it for other purposes in his recently published work, [Gödel [1931]] . . . ." See Feferman [1984], Murawski [1998], Krajewski [2004], and Woleński [2005] for more on Gödel and Tarski vis-á-vis truth.

undefinability of arithmetical truth in arithmetic already in the summer of 1930.[6] In a letter to Zermelo of 12 October 1931 Gödel[7] pointed out that the undefinability of truth leads to a quick proof of incompleteness: The class of provable formulas *is* definable and the class of true formulas is not, and so there must be a true but unprovable formula. Gödel also cited his [1931] footnote 48a, and this suggests that he himself invested it with much significance.[8]

Higher-order satisfaction is particularly relevant both for footnote 48a and the [1932] abstract. Rudolf Carnap at this time was working on his *Logical Syntax of Language*, and in a manuscript attempted a definition of "analyticity" for a language that subsumed the theory of types. Working upward, he provided an adequate definition of truth for first-order arithmetic. In a letter to Carnap of 11 September 1932 Gödel[9] pointed out however that Carnap's attempted recursive definition for second-order formulas contained a circularity. Gödel wrote:

> . . . this error may only be avoided by regarding the domain of the function variables not as the predicates of a definite language, but rather as all sets and relations whatever.[1] On the basis of this idea, in the second part of my work [1931] I will give a definition for "truth", and I am of the opinion that the matter may not be done otherwise . . . .
>
> [1]This doesn't necessarily involve a Platonistic standpoint, for I assert only that this definition (for "analytic") be carried out within a definite language in which one already has the concepts "set" and "relation".

The semantic definition of second-order truth requires "all sets and relations whatever" and must be carried out where one "already has the concepts 'set' and 'relation' ".[10]

[6]Wang [1996, p. 82] reports Gödel as having conveyed the following: Gödel began work on Hilbert's problem to establish the consistency of analysis in the summer of 1930. Gödel quickly distinguished two problems: to establish the consistency of analysis relative to number theory, and to establish the consistency of number theory relative to finitary number theory. For the first problem, Gödel found that he had to rely on the concept of *truth* for number theory, not just the consistency of a formal system for it, and this soon led him to establish the undefinability of truth. The second problem led, of course, to the incompleteness theorems. Note here that Gödel had already focused on establishing *relative* consistency results.

[7]See Feferman and Dawson, Jr. [2003b, pp. 423ff].

[8]In a letter to Gödel of 21 September 1931 Zermelo (see Feferman and Dawson, Jr. [2003b, pp. 420ff]) had actually given the argument for the undefinability of arithmetical truth in arithmetic, thinking that he had found a contradiction in Gödel [1931] whereas he had only conflated truth with provability. This followed the one meeting between Zermelo and Gödel, for which see Kanamori [2004, §7].

[9]See Feferman and Dawson, Jr. [2003a, p. 347].

[10]In a reply of 25 September 1932 to Gödel, Carnap (see Feferman and Dawson, Jr. [2003a, p. 351]) seems somewhat the foil when he asked how this last is to be understood, and further: "Can you define the concept 'set' within a definite formalized semantics?"

A succeeding letter of 28 November 1932 from Gödel to Carnap elaborated on Gödel's footnote 48a.[11] Gödel never actually wrote a Part II to his [1931] and laconically admitted in the letter that such a sequel "exists only in the realm of ideas". Gödel then clarified how the addition of an infinite type ω to the [1931] system P would render provable the unprovable propositions he had constructed—specifically since a truth definition can now be provided. Significantly, Gödel wrote however:

> ... the interest of this definition does not lie in a clarification of the concept 'analytic' since one employs in it the concepts 'arbitrary sets', etc., which are just as problematic. Rather I formulate it only for the following reason: with its help one can show that undecidable sentences become decidable in systems which ascend farther in the sequence of types.

The definition of truth is not itself clarificatory, but it does serve a *mathematical* end.

Tarski, of course, *did* put much store in his systematic definition of truth for formal languages, and Carnap would be much influenced by Tarski's work on truth. Despite their contrasting attitudes toward truth, Gödel's and Tarski's approaches had similarities. Tarski's [1933][1935] undefinability of truth result is couched in terms of languages having "infinite order", analogous to Gödel's [1931] system P having infinite types, and Gödel's infinite type ω is analogous to Tarski's "metalanguage". In a postscript in his [1935, p. 194, n. 108], Tarski acknowledged Gödel's footnote 48a.

In a lecture [1933] Gödel expanded on the themes of our quoted passages. He propounded the axiomatic set theory "as presented by Zermelo, Fraenkel, and von Neumann" as "a natural generalization of the [simple] theory of types, or rather, what becomes of the theory of types if certain superfluous restrictions are removed." First, instead of having separate types with sets of type $n+1$ consisting purely of sets of type n, sets can be *cumulative* in the sense that sets of type n can consist of sets of *all* lower types. If S_n is the collection of sets of type n, then: S_0 is the type of the individuals, and recursively, $S_{n+1} = S_n \cup \{X \mid X \subseteq S_n\}$. Second, the process can be continued into the transfinite, starting with the cumulation $S_\omega = \bigcup_n S_n$, proceeding through successor stages as before, and taking unions at limit stages. Gödel [1933, p. 46] again credited Hilbert for opening the door to the formation of types beyond the finite types. As for how far this cumulative hierarchy of sets is to continue, the "first two or three types already suffice to define very large ordinals" ([1933, p. 47]) which can then serve to index the process, and so on, in an "autonomous progression" in later terminology. In a prophetic remark for set theory and new axioms, Gödel observed: "We set out to find a formal system for mathematics and instead of that found an infinity of

[11] See Feferman and Dawson, Jr. [2003a, p. 355].

systems, and whichever system you choose out of this infinity, there is one more comprehensive, i.e., one whose axioms are stronger." Further echoing the quoted [1932] passage Gödel [1933, p. 48] noted that for any formal system S there is in fact an arithmetical proposition that cannot be proved in S, unless S is inconsistent. Moreover, if S is based on the theory of types, this arithmetical proposition becomes provable if to S is adjoined "the next higher type and the axioms concerning it."

Gödel's approach to set theory, with its emphasis on hierarchical truth, should be set into the context of the axiomatic development of the subject.[12] Zermelo [1908] had provided the initial axiomatization of "the set theory of Cantor and Dedekind", with characteristic axioms Separation, Infinity, Power Set, and of course, Choice. Work most substantially of John von Neumann [1923][1928] on ordinals led to the incorporation of Cantor's transfinite numbers as now the ordinals and the axiom schema of Replacement for the formalization of transfinite recursion. Von Neumann [1929] also formulated the axiom of Foundation, that every set is well-founded, and defined the *cumulative hierarchy* in his system via transfinite recursion: The axiom entails that the universe V of sets is globally structured through a stratification into cumulative "ranks" V_α, where with $\mathcal{P}(X) = \{Y \mid Y \subseteq X\}$ denoting the power set of X,

$$V_0 = \emptyset;\ V_{\alpha+1} = \mathcal{P}(V_\alpha);\ V_\delta = \bigcup_{\alpha<\delta} V_\alpha \text{ for limit ordinals } \delta\,;$$

and

$$V = \bigcup_\alpha V_\alpha\,.$$

Zermelo in his remarkable [1930] subsequently provided his final axiomatization of set theory, proceeding in a second-order context and incorporating both Replacement (which subsumes Separation) and Foundation. These axioms rounded out but also focused the notion of set, with the first providing the means for transfinite recursion and induction and the second making possible the application of those methods to get results about *all* sets. Gödel's coming work would itself amount to a full embrace of Replacement and Foundation but also first-order definability, which would vitalize the earlier initiative of Skolem [1923] to establish set theory on the basis of first-order logic.[13] The now standard axiomatization ZFC is essentially the first-order version of the Zermelo [1930] axiomatization, and ZF is ZFC without AC.

§2. The constructible universe L. Set theory was launched on an independent course as a distinctive field of mathematics by Gödel's formulation of the

[12] For a fuller account documenting the contributions of many, see Kanamori [1996].

[13] See §6 for a comparative analysis of the approaches of Zermelo and Gödel.

class L of *constructible* sets through which he established the relative consistency of AC in mid-1935 and CH in mid-1937.[14] In his first announcement, communicated 9 November 1938, Gödel [1938] wrote:

> "[The] 'constructible' sets are defined to be those sets which can be obtained by Russell's ramified hierarchy of types, if extended to include transfinite orders. The extension to transfinite orders has the consequence that the model satisfies the impredicative axioms of set theory, because an axiom of reducibility can be proved for sufficiently high orders."

This points to two major features of the construction of L:

(i) Gödel had refined the cumulative hierarchy of sets described in his 1933 lecture to a hierarchy of *definable* sets which is analogous to the orders of Russell's *ramified* theory. Despite the broad trend in mathematical logic away from Russell's intensional intricacies and toward versions of the simple theory of types, Gödel had assimilated the ramified theory and its motivations as of consequence and now put the theory to a new use, infusing its intensional character into an extensional context.

(ii) Gödel continued the indexing of the hierarchy through *all* the ordinals as given beforehand to get a class model of set theory and thereby to achieve *relative* consistency results. His earlier [1933, p. 47] idea of using large ordinals defined in low types for further indexing in a bootstrapping process would not suffice. That "an axiom of reducibility can be proved for sufficiently high orders" is an opaque allusion to how Russell's problematic axiom would be rectified in the consistency proof of CH (see §3) and more broadly to how the axiom of Replacement provided for new sets and enough ordinals.[15] Von Neumann's ordinals would be the spine for a thin hierarchy of sets, and this would be the key to both the AC and CH results.

In a brief account [1939b] Gödel informally presented L much as is done today: For any set x let $\operatorname{def}(x)$ denote the collection of subsets of x definable over $\langle x, \in\rangle$ via a first-order formula allowing parameters from x. Then define

$$L_0 = \emptyset;\ L_{\alpha+1} = \operatorname{def}(L_\alpha),\ L_\delta = \textstyle\bigcup_{\alpha<\delta} L_\alpha \text{ for limit ordinals } \delta;$$

and the *constructible universe*

$$L = \textstyle\bigcup_\alpha L_\alpha\,.$$

[14] See Dawson, Jr. [1997, pp. 108, 122]; in one of Gödel's *Arbeitshefte* there is an indication that he established the relative consistency of CH in the night of 14–15 June 1937.

[15] Wang [1981, p. 129], reporting on conversations with Gödel in 1976, wrote how he "spoke of experimenting with more and more complex constructions [of ordinals for indexing] for some extended period somewhere between 1930 and 1935." Kreisel [1980, pp. 193, 196] wrote of Gödel's "reservations" about Replacement which initially held him back from considering all the ordinals as being given beforehand.

Toward the end Gödel [1939b, p. 31] pointed out that L "can be defined and its theory developed in the formal systems of set theory themselves." This is a remarkable understatement of arguably the central feature of the construction of L:

(iii) L is a class definable in set theory via a transfinite recursion that could be based on the formalizability of def(x), the definability of definability. Gödel had not embraced the definition of truth as itself clarificatory,[16] but through his work he in effect drew it into mathematics to a new mathematical end. Though understated in Gödel's writing, his great achievement here as in arithmetic is the submergence of metamathematical notions into mathematics.

In the proof of the incompleteness theorem, Gödel had encoded provability—syntax—and played on the interplay between truth and definability. Gödel now encoded satisfaction—semantics—with the room offered by the transfinite indexing, making truth, now definable for levels, part of the formalism and part of the subject matter. In modern parlance, an *inner model* of ZFC is a transitive (definable) class containing all the ordinals such that, with membership and quantification restricted to it, the class satisfies each axiom of ZFC. Gödel in effect argued in ZF to show that L is an inner model of ZFC, and moreover that L satisfies CH. He thus established the relative consistency Con(ZF) implies Con(ZFC + CH). In what follows, we describe his proofs that L is an inner model of ZFC and in §3 that L satisfies CH.

In his sketch [1939b] Gödel simply argued for the ZFC axioms holding in L as evident from the construction, with the extent of the ordinals and the sets provided by def(x) sufficient to establish Replacement in L. Only at the end when he was attending to formalization did he allude to the central issue of relativization. For here and later, recall that for a formula φ and classes C and M, φ^M and C^M denote the relativizations to M of φ and C respectively, i.e., φ^M denotes φ but with the quantifiers restricted to the elements of M, and C^M denotes the class defined by the relativization to M of a defining formula for C. Gödel's [1939b] arguments for relative consistency amount to establishing φ^L as theorems of set theory for various φ starting with the axioms of set theory themselves, and could only work if $\mathrm{def}^L(x) = \mathrm{def}(x)$ for $x \in L$. This *absoluteness of first-order definability* is central to the proof if L is to be formally defined via the def(x) operation, but notably Gödel himself would never establish this absoluteness explicitly, preferring in his one rigorous published exposition of L to take an approach that avoids def(x) altogether.

In his monograph [1940a], based on 1938 lectures, Gödel provided a specific, formal presentation of L in a class-set theory developed by Paul Bernays

[16]See the last displayed passage in §1.

[1937], a theory based in turn on a theory of von Neumann [1925]. First, Gödel carried out a paradigmatic development of "abstract" set theory through the ordinals and cardinals with features that have now become common fare, like his particular well-ordering of pairs of ordinals.[17] Gödel then used eight binary operations, producing new classes from old, to generate L set by set via transfinite recursion. This veritable "Gödel numbering" with ordinals bypassed the $\text{def}(x)$ operation and made evident certain aspects of L. Since there is a direct, definable well-ordering of L, choice functions abound in L, and AC holds there.

Much of the analysis of L would have to be devoted to verifying Replacement or at least Separation there, this requiring an analysis of the first-order formalization of set properties. It has sometimes been casually asserted that Gödel [1940a] through his eight operations provided a finite axiomatization of Separation, but this cannot be done. Through closure under the operations one does get Separation for *bounded* formulas, i.e., those formulas all of whose quantifiers can be rendered as $\forall x \in y$ and $\exists x \in y$. Gödel established using Replacement (in V) that for any set $x \subseteq L$, there is a $y \in L$ such that $x \subseteq y$ (9.63 of [1940a]). He then established that a wide range of classes $C \subseteq L$ satisfy the condition that for any $x \in L$, $x \cap C \in L$, that C is "amenable" in later terminology. With this, he established σ^L for every axiom σ of ZFC, the relativized instances of Replacement being the most crucial to confirm.[18] Having bypassed $\text{def}(x)$, this argumentation makes no appeal to absoluteness.

§3. Consistency of the Continuum Hypothesis. Gödel's proof that L satisfies CH consisted of two separate parts. He established the implication $V = L \rightarrow \text{CH}$, and, in order to apply this implication within L, the absoluteness $L^L = L$ to establish the desired $(\text{CH})^L$. That $V = L \rightarrow \text{CH}$ established a connection between two quite non-absolute concepts, the power set and successor cardinality of an infinite set, and the absoluteness $L^L = L$ effected the requisite relativization. That $L^L = L$ had been asserted in his first announcement [1938], and follows directly from $\text{def}^L(x) = \text{def}(x)$ for $x \in L$, which was broached in the sketch [1939b]. In [1940a], his approach to $L^L = L$ was rather through the evident absoluteness of the eight generating operations which in particular entailed that being a (von Neumann) ordinal is absolute and ensured the internal integrity of the generation of L. There is a nice resonance here with Gödel [1931], in that there he had catalogued a series of functions to be primitive recursive whereas now he catalogued a series of set-theoretic operations to be absolute—the submergence of provability (syntax) for arithmetic evolved to the submergence of definability

[17] In footnote 14 added in a 1951 printing of his [1940a] Gödel (see Feferman [1990, p. 54]) even used the device later attributed to Dana S. Scott [1955] for reducing classes to sets by restricting to members of lowest rank.

[18] Jech [2002, §13] presents a modern version of Gödel's argument.

(semantics) for set theory. The argument in fact shows that for any inner model M of ZFC, $L^M = L$. Decades later many inner models based on first-order definability would be investigated for which absoluteness considerations would be pivotal, and Gödel had formulated the canonical inner model.

Gödel's argument for $V = L \rightarrow$ CH rests, as he himself wrote in a brief summary [1939a], on "a generalization of Skolem's method for constructing enumerable models." This was the first significant use of Skolem functions since Skolem's own [1920] to establish the Löwenheim–Skolem theorem. Gödel [1939b] specifically established:

For infinite α, every constructible subset of L_α

belongs to some L_β for a β of the same cardinality as α. $(*)$

It is straightforward to show that for infinite α, L_α has the same cardinality as that of α. It follows from $(*)$ that in L, the power set of L_ω is included in L_{ω_1}, and so CH follows. (Gödel emphasized the Generalized Continuum Hypothesis (GCH), that $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all α, and $V = L \rightarrow$ GCH follows by analogous reasoning.) Gödel [1939b] proved $(*)$ for an $X \subseteq L_\alpha$ such that $X \in L$ by getting a set $M \subseteq L$ containing X and sufficiently many ordinals and definable sets so that M will be isomorphic to some L_β, the construction of M ensuring that β has the same cardinality as α. Gödel's approach to M, different from the usual approach taken nowadays, can be seen as proceeding through layers defined recursively, a new layer being defined via closure according to new Skolem functions and ordinals based on the preceding layer. This was indeed a "generalization of Skolem's method", being an *iterative* application of Skolem closures. M having been sufficiently bolstered, Gödel then confirmed that M is isomorphic with respect to $\in$ to some L_β, making the first use of the now familiar Mostowski transitive collapse.

Gödel in his monograph [1940a], having proceeded without def(x), formally carried out his [1939b] argument in terms of his eight operations, and this had the effect of obscuring the Skolem definability and closure. There is, however, an economy of means that can be seen from Gödel [1940a]: The arguments there demonstrated that absoluteness is not necessary to establish either that L is an inner model of ZFC or that $V = L \rightarrow$ CH; absoluteness is only necessary where it is intrinsic, to establish $L^L = L$.

Until the 1960s accounts of L dutifully followed Gödel's [1940a], presentation, and papers generally in axiomatic set theory often used and referred to Gödel's specific listing and grouping of his class-set axioms. However, modern expositions of L proceed in ZFC with the direct formalization of def(x), first formulating satisfaction-in-a-structure and coding this in set theory. They then establish Replacement or Separation in L by appealing to an L analogue of the ZF Reflection Principle, drawn from Richard

Montague [1961, p. 99] and Azriel Levy [1960, p. 234].[19] Moreover, they establish $V = L \to \mathrm{CH}$ via some version of the *Condensation Lemma*: If δ is a limit ordinal and X is an elementary substructure of L_δ, then there is a β such that X is isomorphic to L_β. Instead of Gödel's hand-over-hand algebraic approach to get $(*)$, one incorporates the satisfaction-in-a-structure relation and takes at least a Σ_1-elementary substructure of an ambient L_δ in a uniform fashion using its Skolem functions. This higher-level approach is indicative of how the satisfaction relation has been assimilated into modern set theory but also of what Gödel's approach had to encompass.

One is left to speculate why, and perhaps to rue that, Gödel did not himself articulate a reflection principle for use in L or some version of the Condensation Lemma based on the model-theoretic satisfaction-in-a-structure relation. The requisite Skolem closure argument would have served as a motivating *entrée* into his [1939b] proof of CH in L. Moreover, this approach would have provided a thematic link to Gödel's later advocacy of the heuristic of reflection, described in §7. Finally, with satisfaction-in-a-structure becoming the basis of model theory after Tarski–Vaught [1957] and the ZF Reflection Principle emerging only through the infusion of model-theoretic methods into set theory around 1960, a fuller embrace by Gödel of the satisfaction relation might have accelerated the process. That infusion was stimulated by Tarski through his students, and this sets in new counterpoint Gödel's indirect engagement with truth and satisfaction.[20]

Gödel's fine grained [1940a] approach made transparent the absoluteness of L without having to confront $\mathrm{def}(x)$, but it also obfuscated the intuitive underpinnings of definability and the historical motivations, and this may have hindered the understanding of L for years. On the other hand, once L became assimilated, Gödel's [1940a] presentation would serve as the direct precursor for Ronald Jensen's [1972] potent and fruitful fine structure theory.

§4. Descriptive set theory results. In his first announcement [1938] Gödel listed together with the Axiom of Choice and the Generalized Continuum Hypothesis two other propositions that hold in L. These were propositions

[19]The principle asserts that for any (first-order) formula $\varphi(v_1, \ldots, v_n)$ in the free variables as displayed and any ordinal β, there is a limit ordinal $\alpha > \beta$ such that for any $x_1, \ldots, x_n \in V_\alpha$ we have $\varphi[x_1, \ldots, x_n]$ *iff* $\varphi^{V_\alpha}[x_1, \ldots, x_n]$, where again φ^M denotes the relativization of the formula φ to M. This principle is equivalent to Replacement and Infinity in the presence of the other ZF axioms.

[20]It is, however, notable that in a seminal paper, Tarski [1931] gave a precise, set-theoretic formulation of the concept of a set of reals being first-order definable in the structure $\langle \mathrm{Reals}, +, \times \rangle$ that bypassed formulating the concept of satisfaction in this structure. Rather, Tarski worked with Boolean combinations and geometric projections. Like Gödel, Tarski at the time worked to dispense with the metamathematical underpinnings. With the development of mathematical logic, we now see results stated there as leading to the decidability of real closed fields via the elimination of quantifiers.

of descriptive set theory, the definability theory of the continuum.[21] To state them in modern terms, we first recall some terminology: With $\mathbb{R}$ the set of real numbers and considering $\mathbb{R}^n$ as a topological space in the usual way, suppose that $Y \subseteq \mathbb{R}^n$. Y is $\mathbf{\Sigma}^1_1$ (*analytic*) *iff* Y is the projection $pB = \{\langle x_1, \dots, x_n \rangle \mid \exists y(\langle x_1, \dots, x_n, y \rangle \in B)\}$ of a Borel subset B of $\mathbb{R}^{n+1}$. (Equivalently, Y is the image under a continuous function of a Borel subset of some $\mathbb{R}^k$.) Y is $\mathbf{\Pi}^1_1$ *iff* $\mathbb{R}^n - Y$ is $\mathbf{\Sigma}^1_1$. Y is $\mathbf{\Sigma}^1_2$ *iff* Y is the projection of a $\mathbf{\Pi}^1_1$ subset of $\mathbb{R}^{n+1}$. Y is $\mathbf{\Pi}^1_2$ *iff* $\mathbb{R}^n - Y$ is $\mathbf{\Sigma}^1_2$. Y is $\mathbf{\Delta}^1_2$ *iff* it is both $\mathbf{\Sigma}^1_2$ and $\mathbf{\Pi}^1_2$. Proceeding thus through finite indices we get the hierarchy of *projective* sets. A set of reals has the *perfect set property* if either it is countable or else has a perfect subset.[22] Gödel's propositions following from $V = L$ can be cast as follows:

(a) There is a $\mathbf{\Delta}^1_2$ set of reals which is not Lebesgue measurable.

(b) There is a $\mathbf{\Pi}^1_1$ set of reals which does not have the perfect set property.

It had been known from Luzin [1914] that every $\mathbf{\Sigma}^1_1$ set is Lebesgue measurable and has the perfect set property, and so (a) and (b) provided an explanation in terms of relative consistency about the lack of progress up the projective hierarchy.

Gödel never again mentioned (a) or (b) in print, and only in an endnote to a 1951 printing of his [1940a] did he describe a relevant result. There, he pointed out that the inherent [1940a] well-ordering of L when restricted to its reals is a $\mathbf{\Sigma}^1_2$ subset of $\mathbb{R}^2$, describing how generally to incorporate his [1940a] development into the definability context of descriptive set theory. When every real is in L, this $\mathbf{\Sigma}^1_2$ well-ordering is $\mathbf{\Delta}^1_2$ and does not satisfy Fubini's Theorem for Lebesgue measurable subsets of the plane, and this is one way to confirm (a). (b) is most often derived indirectly; what may have been Gödel's original argument is given in Kanamori [2003, p. 170].

Correspondence with von Neumann casts some light here. In a letter to von Neumann of 12 September 1938 Gödel[23] pointed out: "The theorem on one-to-one continuous images of [$\mathbf{\Pi}^1_1$] sets, which we had discussed at our last meeting, turned out to be false (refuted by Mazurkiewicz in Fund[amenta Mathematicae] 10). … I now even have some results in the opposite direction … " What was at issue here were images under *one-to-one* continuous functions. Gödel had been working on ongoing mathematics and would use L to address a mathematical question by giving a negative consistency result as per the axioms of set theory—a new kind of impossibility result.

With the reconstrual of projections as continuous real functions, the $\mathbf{\Sigma}^1_2$ sets are exactly the sets that are the continuous images of $\mathbf{\Pi}^1_1$ sets. Noting

[21] See Moschovakis [1980] or Kanamori [2003] for the concepts and terminology of descriptive set theory. See Kanamori [1995] for the emergence of descriptive set theory.

[22] A set of reals is perfect if it is non-empty, closed, and has no isolated points.

[23] See Feferman and Dawson, Jr. [2003b, p. 361].

that Sierpiński had asked whether the *one-to-one* continuous image of any $\mathbf{\Pi}^1_1$ set is again $\mathbf{\Pi}^1_1$, Mazurkiewicz [1927] had observed that this was not so by showing that the difference of any two $\mathbf{\Sigma}^1_1$ sets is the one-to-one continuous image of a $\mathbf{\Pi}^1_1$ set. In his letter to von Neumann, Gödel proceeded to announce the consistency of (b) and the version of (a) asserting that there is a non-measurable (and hence not $\mathbf{\Pi}^1_1$) one-to-one continuous image of a $\mathbf{\Pi}^1_1$ set. With the recasting of continuous functions as projections, his actual statement of (a) in [1938], communicated two months later on 9 November 1938, was in the stringent and telling form: There is a non-measurable set such that both it and its complement are *one-to-one* projections of $\mathbf{\Pi}^1_1$ subsets of $\mathbb{R}^2$. Gödel was focused on one-to-one images, and one can reconstruct this consistency result with L.[24]

In a letter to Gödel of 28 February 1939 von Neumann (see Feferman and Dawson, Jr. [2003b, p. 363]) brought to his attention the paper of Motokiti Kondô [1939]. For $A, B \subseteq \mathbb{R}^2$, A is *uniformized* by B *iff* $B \subseteq A$ and $\forall x(\exists y(\langle x, y\rangle \in A) \leftrightarrow \exists! y(\langle x, y\rangle \in B))$. Kondô [1939] had established the culminating result of the early, classical period of descriptive set theory, that every $\mathbf{\Pi}^1_1$ subset of $\mathbb{R}^2$ can be uniformized by a $\mathbf{\Pi}^1_1$ set. With this it was immediate that *every* $\mathbf{\Sigma}^1_2$ set is the *one-to-one* continuous image of a $\mathbf{\Pi}^1_1$ set, and (a) as stated above is indeed equivalent to Gödel's original [1938] form. In a letter to von Neumann of 20 March 1939 Gödel[25] wrote: "The result of Kondô is of great interest to me and will definitely allow an important simplification in the consistency proof of [(a)] and [(b)] of the attached offprint."[26]

Gödel's results (a) and (b) can be put into a broad historical context. Cantor's early preoccupation was with sets of reals and the like, and substantially motivated by his CH he both developed the transfinite numbers and investigated topological properties of sets of reals. In particular, he established that the closed sets have the perfect set property and so "satisfy the CH" since perfect sets have the cardinality of the continuum. Zermelo developed *abstract* set theory, with $\in$ having no privileged interpretation and sets regulated and generated by axioms.[27] In the first decades of the 20th

[24]For example, one can apply the idea used in Kanamori [2003, p. 170].

[25]See Feferman and Dawson, Jr. [2003b, p. 365].

[26]All this to and fro tends to undermine the eye-catching remark of Kreisel [1980, p. 197] that: " ... according to Gödel's notes, not he, but S. Ulam, steeped in the Polish tradition of descriptive set theory, noticed that the definition of the well-ordering ... of subsets of ω was so simple that it supplied a non-measurable PCA [i.e., $\mathbf{\Sigma}^1_2$] set of real numbers ... ."

[27]See Kanamori [2004] for Zermelo and set theory. Concerning "abstract", Fraenkel in his text *Abstract Set Theory* [1953] distinguished between *abstract sets* (the nature of whose elements are not of concern) and *sets of points* (typically numbers). In the early years "general set theory" was also sometimes used with connotations similar to "abstract set theory", though Zermelo himself consistently used "general set theory" to refer to axiomatic set theory without Infinity. The latter-day Skolem [1962] was still entitled *Abstract Set Theory*.

Century *descriptive* set theory carried forth the investigation of sets of reals through the Borel and analytic sets into the projective sets, while in abstract set theory Cantor's transfinite numbers were incorporated into the axiomatic framework by von Neumann with his ordinals. Then formal definability was brought into descriptive set theory by Tarski [1931], which before his well-known paper [1933] on truth dealt with the concept of a first-order definable set of reals, and by Kuratowski–Tarski [1931] and Kuratowski [1931], which pursued the basic connection between existential number quantifiers and countable unions and between existential real quantifiers and projection and used these "logical symbols" to aid in the classification of sets in the Borel and projective hierarchies. Gödel in his monograph [1940a, p. 3] developed "abstract" set theory, and in that 1951 endnote started *ab initio* to correlate definability in L with formal definability in descriptive set theory. Gödel's results (a) and (b) constitute the first real synthesis of abstract and descriptive set theory, in that the axiomatic framework is brought to bear on the investigation of definable sets of reals.

§5. L **through the lectures.** Gödel's posthumously published lectures [1939c] and [1940b] provide considerable insight into his motivations and development of L. Both Hilbert and Russell loom large in Gödel's lecture [1939c], given at Hilbert's Göttingen on 15 December 1939. Gödel recalled at length Hilbert's previous work [1926] on CH and cast his own as an analogical development, one leading however to the constructible sets as a model for set theory. Hilbert [1926] apparently thought that if he could show that from any given formalized putative disproof of CH, he could prove CH, then CH would have been established. At best, Hilbert's argument could only establish the relative *consistency* of CH; this was evident to Gödel, who unlike Hilbert saw the distinction between truth and consistency clearly and wrote [1939c, p. 129] "the first to outline a *program* for a consistency proof of the continuum hypothesis was *Hilbert*." For Hilbert, any disproof of CH would have to make use of number-theoretic functions whose definitions in his system needed his ε-symbol, his well-known device for abstracting quantification. He thus set out to replace the use of such functions by functions defined instead by transfinite recursion through the countable ordinals and via recursively defined higher-type functionals. The influence of Russell's ramified hierarchy is discernible here both in the preoccupation with definability and with the introduction of a type hierarchy, albeit one extended into the transfinite. Finally, Hilbert's scheme rested on establishing a bijection between such definitions and the countable ordinals to establish CH.

Gödel started his description of L by recalling two main lemmas in Hilbert's argument and casting two main features of L in analogous fashion. Contrasting his approach with Hilbert's however, Gödel [1939c, p. 131] emphasized about L that "*the model ... is by no means finitary*; in other words, the transfinite and impredicative procedures of set theory enter into its definition in an essential way, and that is the reason why one obtains only a relative consistency proof [of CH] ... " Gödel then pointed out a crucial property of L to which there was no Hilbertian counterpart, that it has "a certain *invariance*" property, i.e., the absoluteness $L^L = L$. To motivate the model Gödel again referred to Russell's ramified theory of types. Gödel first described what amounts to the orders of that theory for the simple situation when the members of a countable collection of real numbers are taken as the individuals and new real numbers are successively defined via quantification over previously defined real numbers, and emphasized that the process can be continued into the transfinite. He [1939c, p. 131] then observed that this procedure can be applied to sets of real numbers and the like, as individuals, and moreover, that one can "intermix" the procedure for the real numbers with the procedure for sets of real numbers "by using in the definition of a real number quantifiers that refer to sets of real numbers, and similarly in still more complicated ways." Gödel called a *constructible* set "the most general [object] that can at all be obtained in this way, where the quantifiers may refer not only to sets of real numbers, but also to sets of sets of real numbers and so on, *ad transfinitum*, and where the indices of iteration ... can also be arbitrary transfinite ordinal numbers." Gödel [1939c, p. 137] considered that although this definition of constructible set might seem at first to be "unbearably complicated", "the *greatest generality yields,* as it so often does, at the same time the *greatest simplicity*." Gödel was picturing Russell's ramified theory of types by first disassociating the types from the orders, with the orders here given through definability and the types represented by real numbers, sets of real numbers, and so forth. Gödel's intermixing then amounted to a recapturing of the complexity of Russell's ramification, the extension of the hierarchy into the transfinite allowing for a new simplicity.

Gödel [1939c, p. 137] went on to describe the universe of set theory, "the objects of which set theory speaks", as falling into "a transfinite sequence of Russellian [simple] types" the cumulative hierarchy of sets that he had described in his [1933]. He then formulated the constructible sets as an analogous hierarchy, the hierarchy of [1939b]. Giving priority to the ordinals, Gödel had introduced transfinite Russellian orders through definability, and the hierarchy of types was spread out across the orders. The jumble of the *Principia Mathematica* had been transfigured into the model L of the constructible universe. Gödel forthwith pointed out a salient difference between the V and the L hierarchies with respect to cardinality: Whereas $|V_{\alpha+1}| > |V_\alpha|$ because of the use of the power set operation, $|L_{\alpha+1}| = |L_\alpha| = |\alpha|$ for infinite α.

In a comment bringing out the intermixing of types and orders, Gödel [1939c, p. 141] pointed out that "there are sets *of lower type* that *can* only *be defined* with the help of *quantifiers for sets of higher type*." Constructible subsets of L_ω will first appear high in the L hierarchy; in terms of the [1933, p. 48] remarks, sets of natural numbers will encode truth propositions about higher L_α's. However, these cannot be arbitrarily high. Gödel [1939c, p. 143] announced the version of $(*)$ (cf. §3) for countable ordinals as the crux of the consistency proof of CH. He subsequently asserted that "this fundamental theorem constitutes the corrected core of the so-called Russellian axiom of reducibility." Thus, Gödel established another connection between L and Russell's ramified theory of types. But while Russell had to *postulate* his axiom of Reducibility for his finite orders, Gödel was able to *prove* an analogous form for his transfinite hierarchy, one that asserts that the types are delimited in the hierarchy of orders. Not only did Gödel resurrect the ramified theory with L, but his transfinite type extension rectified Russell's ill-fated axiom. Reflecting a remark from [1931] quoted in §1 about the axiom of Reducibility as "the comprehension axiom of set theory", Gödel wrote [1939c, p. 145]:

> This character of the fundamental theorem as an axiom of reducibility is also the reason why *the axioms of classical* mathematics hold for the model of the constructible sets. For after all, as Russell showed, the axioms of reducibility, infinity and choice are the only axioms of classical mathematics that do not have [a] tautological character. To be sure, one must observe that the axiom of reducibility appears in different mathematical systems under different names and in different forms, for example, in Zermelo's system of set theory as the axiom of separation, in Hilbert's systems in the form of recursion axioms, and so on.

This passage shows Gödel to be holding a remarkably synthetic, unitary view, viewing as he does Russell's axiom of Reducibility, Zermelo's Separation axiom, and Hilbert's [1926] recursion axioms all as one. Actually, $(*)$ as such is not necessary to establish that L is a model of set theory; it is sufficient that for any α, the constructible subsets of L_α all belong to some L_β and for this one only needs the full extent of the ordinals as bolstered by Replacement. That $(*)$ is sufficient but separate is acknowledged by Gödel when he next wrote: "Now the axiom of reducibility holds for the constructible sets on the basis of the fundamental theorem ... " Thus, it is more proper to regard Reducibility, Replacement, and the Reflection Principle (cf. end of §3) all as one, and the thrust of Gödel's comments on Reducibility are more in this direction.

Gödel in his lecture did not detail the proof of $L^L = L$, mentioning [1939c, p. 145] only that "an essential point in it is that the notion of ordinal number is absolute: that is, ordinal number in the model of the constructible sets means the same as ordinal number itself." He then launched into a detailed account of the proof of the "fundamental theorem", i.e., $(*)$ for countable ordinals, the proof being the one sketched in [1939b]. This lecture of Gödel's is a remarkably clear presentation of both the mathematical and historical development of L, and had it become widely accessible together with his [1940a], it would no doubt have accelerated the assimilation of L.

Hilbert and Russell also figure prominently in a later lecture [1940b] on CH given on 15 November 1940 at Brown University, of which we mainly describe the new ground covered. Gödel began by announcing that he had "succeeded in giving the [consistency] proof a new shape which makes it somewhat similar" to Hilbert's [1926] attempt and proposed to sketch the new proof, considering it "perhaps the most perspicuous". First however, Gödel described the issues involved in general terms and reviewed the $\mathrm{def}(x)$ construction of L. Once again he emphasized that his argument for showing that CH holds in L proves an axiom of reducibility, this time putting more stress on Separation [1940b, p. 178]: "... it is not surprising that the axioms of set theory hold for the constructible sets, because the axiom of reducibility or its equivalents, e.g., Zermelo's Aussonderungsaxiom [Separation], is really the only essential axiom of set theory." Gödel then turned to his new approach and introduced the concept of a relation being "recursive of order α" for ordinals α. This concept is a generalization of the notion of definability, a generalization obtained by essentially interweaving the operation $\mathrm{def}(x)$ with a recursion scheme akin to Hilbert's for his [1926] hierarchy of functionals.[28] As Gödel [1940b, p. 180] wrote: "The difference between this notion of recursiveness and the one that Hilbert seems to have had in mind is chiefly that I allow quantifiers to occur in the definiens." This, of course, is a crucial difference, and having separated out arithmetical aspects of definability à la Hilbert, Gödel [1940b, p. 181] because of the quantifiers had to face head on "defining recursively the metamathematical notion of truth" à la Tarski. This 1940 juncture is arguably when Gödel came closest, having never written that part II to his [1931], to describing what could have been its contents:

> Now this metamathematical notion of truth, i.e., the class of numbers of truth propositions, can be defined by a method similar to the one which Tarski applied for the system of *Principia mathematica*. The point is to well-order all propositions of our domain in such a manner that the truth of each depends in a precisely describable manner on the truth of some of the preceding; this gives then the desired recursive definition.

Using the new concept of recursiveness—better, new concept of definability—Gödel gave a model of Russell's *Principia*, construed as his [1931], system P, in which CH holds. The types of this model were essentially coded versions of $L_{\omega_{n+1}} - L_{\omega_n}$. Echoing his [1931], footnote 48a, Gödel [1940a, p. 184] subsequently wrote:

[28]As analyzed by Solovay in his introductory notes (cf. Feferman [1995, p. 122]), for $\alpha > \omega$, a relation on α is recursive of order α exactly when it appears in $L_{\alpha \cdot \omega}$.

> You know every formal system is incomplete in the sense that it can be enlarged by new axioms which have approximately the same degree of evidence as the original axioms. The most general way of accomplishing these enlargements is by adjoining higher types, e.g., the type ω for the system of *Principia mathematica*. But you will see that my proof goes through for systems of arbitrarily high type.

However high a transfinite type that one wanted to include, one can similarly establish the relative consistency of CH in the corresponding "inner model".

A coda, returning to truth: Years later, in a letter to Hao Wang of 7 March 1968 Gödel[29] wrote, in implicit criticism of Hilbert:

> ... there was a special obstacle which *really* made it *practically impossible* for constructivists to discover my consistency proof. It is the fact that the ramified hierarchy, which had been invented *expressly for constructive purposes,* had to be used in an *entirely nonconstructive way.* A similar remark applies to the concept of mathematical truth, where formalists considered formal demonstrability to be an *analysis* of the concept of mathematical truth and, therefore, were of course not in a position to *distinguish* the two.

Wang [1996, pp. 250ff] described how Gödel in January 1972 retrospectively contrasted Hilbert's approach to CH with his.

§**6. Set theory transformed.** Gödel with L brought into set theory a *method* of construction and argument and thereby affirmed several features of its axiomatic presentation. First, Gödel showed how first-order definability can be formalized and used in a transfinite recursive construction to establish striking new mathematical results. This significantly contributed to a lasting ascendancy for first-order logic, which beyond its *sufficiency* as a logical framework for mathematics was seen to have considerable *operational efficacy*. Gödel's construction moreover buttressed the incorporation of Replacement and Foundation into set theory. Replacement was immanent in the arbitrary extent of the ordinals for the indexing of L and in its formal construction via transfinite recursion. In his analysis of Russell's mathematical logic Gödel [1944, p. 147] again wrote about how with L he had proved an axiom of reducibility, and in fact that " ... all impredicativities are reduced to one special kind, namely the existence of certain large ordinal numbers (or well-ordered sets) and the validity of recursive reasoning for them." As for Foundation, underlying the construction was the well-foundedness of sets. Gödel in a footnote to his account [1939b, fn12] wrote about his axiom A, i.e., $V = L$: "In order to give A an intuitive

[29]See Feferman and Dawson, Jr. [2003b, p. 404].

meaning, one has to understand by 'sets' all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders)." Gödel [1947, pp. 518ff] later wrote:[30]

> ... there exists a satisfactory foundation of Cantor's set theory in its whole original extent, namely axiomatics of set theory, under which the logical system of *Principia mathematica* (in a suitable interpretation) may be subsumed.
>
> It might at first seem that the set-theoretical paradoxes would stand in the way of such an undertaking, but closer examination shows that they cause no trouble at all. They are a very serious problem, but not for Cantor's set theory.... This concept of set ... according to which a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of", and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naïve" and uncritical working with this concept of set has so far proved completely self-consistent.

A new emphasis here is on the inherent consistency of the cumulative hierarchy stratification, which, to emphasize, is provided by the axioms, most saliently Foundation interacting with Replacement, Power Set, and Union.

The approaches of Gödel and of Zermelo [1930] (mentioned in §1) to set theory merit comparison with respect to the emergence of the cumulative hierarchy view, the focus on models of set theory, and subsequent influence.[31] Zermelo had first adopted Foundation, thereby promoting the cumulative hierarchy view of sets, and posited an endless procession of models of his axioms of form V_κ for inaccessible cardinals[32] κ with one model a set in the next. Both Zermelo and Gödel advocated direct transfinite reasoning, with Zermelo proceeding in an avowedly second-order axiomatic context and Gödel formalizing first-order definability in his transfinite extension of the theory of types. Gödel came close to Zermelo [1930] in his informal sketch [1939b] about L when he stated his relative consistency results in terms of the axioms of Zermelo [1908] as rendered in first-order logic and asserted that L_Ω, where Ω is "the first inaccessible number", is a model of Zermelo's axioms together with Replacement. Also, making his only explicit reference to Zermelo [1930], Gödel [1947, p. 520] later gave the existence of inaccessible cardinals as the simplest example of an axiom that asserts still further iterations of the 'set of' operation

[30] Here the footnotes to the text are excised.

[31] Kreisel [1980] draws this comparison for didactic purposes.

[32] An uncountable cardinal κ is *inaccessible* if κ is a regular cardinal, i.e., if $\alpha < \kappa$ and $F : \alpha \to \kappa$, then $\bigcup F``\alpha < \kappa$, and κ is a strong limit cardinal, i.e., if $\beta < \kappa$, then $2^\beta < \kappa$.

and can supplement the axioms of set theory without arbitrariness.[33] Beyond the imprint on Gödel himself, which could be regarded as significant, Zermelo [1930] seemed to have had little influence on the further development of set theory, presumably because of its second-order lens and its lack of rigorous detail and attention to relativism.[34] On the other hand, Gödel's work with L with its incisive analysis and use of first-order definability was readily recognized as a signal advance. Issues about consistency, truth, and definability were brought to the forefront, and the CH result established the mathematical importance of a hierarchical analysis. As the construction of L was gradually digested, the sense it promoted of a cumulative hierarchy reverberated to become the basic picture of the universe of sets.

How Gödel transformed set theory can be broadly cast as follows: On the larger stage, from the time of Cantor, sets began making their way into topology, algebra, and analysis so that by the time of Gödel, they were fairly entrenched in the structure and language of mathematics. But how were sets viewed among set *theorists*, those investigating sets as such? Before Gödel, the main concerns were what sets *are* and how sets and their axioms can serve as a reductive basis for mathematics. Even today, those preoccupied with ontology, questions of mathematical existence, focus mostly upon the set theory of the early period. After Gödel, the main concerns became what sets *do* and how set theory is to advance as an autonomous field of mathematics. The cumulative hierarchy picture was in place as subject matter, and the metamathematical methods of first-order logic mediated the subject. There was a decided shift toward epistemological questions, e.g., what can be proved about sets and on what basis.

§7. **Truth and new axioms.** A pivotal figure Gödel, what was his own stance? What he *said* would align him more with his predecessors, but what he *did* would lead to the development of methods and models. In a critical analysis [1944] of Russell's mathematical logic, a popular discussion [1947] of Cantor's continuum problem, and subsequent lectures and correspondence, Gödel articulated his philosophy of "conceptual realism" about mathematics. He espoused a staunchly objective "concept of set" according to which the axioms of set theory are true and are descriptive of an objective reality schematized by the cumulative hierarchy. Be that as it may, his actual mathematical work laid the groundwork for the development of a range of

[33]Gödel referenced Zermelo [1930] after writing: "[This] axiom, roughly speaking, means nothing else but that the totality of sets obtainable by exclusive use of the processes of formation of sets expressed in the other axioms forms again a set (and, therefore, a new basis for a further application of these processes)." This was just what Zermelo had emphasized; for Gödel there would also be the overlay of truth in the "next higher system".

[34]For the record, Kreisel [1980, p. 193] wrote that Zermelo's paper "made little impression" but adduced historically peculiar reasons.

models and axioms for set theory. Already in 1942 Gödel worked out for himself a possible model for the negation of AC in the framework of type theory.[35] In his steady intellectual development Gödel would continue to pursue the distinction between truth and provability into the higher reaches of set theory.

In oral, necessarily brief remarks at a conference Gödel [1946] made substantial mathematical suggestions that newly engaged truth in terms of absoluteness and with concepts involving the heuristic of *reflection*. Pursuing his "next higher system" theme Gödel explored possible absolute notions of demonstrability and definability, those not dependent on any particular formalism. For absolute demonstrability, Gödel again pointed out how formalisms can be transcended and the process iterated into the transfinite. And recalling his remarks about L, he pointed out that while no one formalism would embrace the entire process, "it could be described and collected in some non-constructible way". Gödel then charted new waters, with remarks having an anticipation in the [1932] passage quoted in §1:

> In set theory, e.g., the successive extensions can most conveniently be represented by stronger and stronger axioms of infinity. It is certainly impossible to give a combinational and decidable characterization of what an axiom of infinity is; but there might exist, e.g., a characterization of the following sort: An axiom of infinity is a proposition which has a certain (decidable) formal structure and which in addition is true. Such a concept of demonstrability might have the required closure property, i.e., the following could be true: Any proof for a set-theoretic theorem in the next higher system above set theory (i.e., any proof involving the concept of truth which I just used) is replaceable by a proof from such an axiom of infinity. It is not impossible that for such a concept of demonstrability some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of sets.

This is a remarkably optimistic statement about the possibility of discovering new "true" axioms that will decide *every* set-theoretic proposition. The engagement with truth has introduced a new element, "strong axioms of infinity", and an argument by reflection: "Any proof for a set-theoretic theorem in the next higher system above set theory", i.e., if the satisfaction relation for V itself were available, "is replaceable by a proof from such an axiom of infinity." There is still an afterglow here from Russell's axiom of Reducibility as filtered through Gödel's work. Reaching further back, there is more resonance with another notion of absoluteness, Cantor's of

[35]See e.g., Dawson, Jr. [1997, p. 160].

the absolutely infinite, or the Absolute.[36] Recast in terms of the cumulative hierarchy, the universe $V = \bigcup_\alpha V_\alpha$ cannot be comprehended, and so any particular property ascribable to it must already be ascribable to some rank V_α, some postulations becoming the strong axioms.

For absolute definability, Gödel pointed out that here also there is a transfinite hierarchy, one of "concepts of definability", and "it is not possible to collect together all these languages in one, as long as you have a finitistic concept of language." Whereas for demonstrability he had envisioned the use of strong axioms of infinity, for definability he turned to expanding the language by allowing constants for *every* ordinal. This is resonant with Gödel's formulation of L in that the main non-constructive feature is the indexing through the ordinals and their arbitrary extent is again brought to the fore and made use of. Gödel [1946, p. 3] made a crucial claim:

> By introducing the notion of truth for this whole transfinite language, i.e., by going over to the next language, you will obtain no new definable sets (although you will obtain new definable properties of sets).

The passages quoted in §1 and the construction of L had featured the introduction of higher types allowing for the definability of new satisfaction relations and hence new definable sets of lower type. Gödel saw that having the satisfaction relation for set theory for the enriched language with constants for every ordinal leads to a closure for definability, "no new definable sets", as separated from truth, "new definable properties of sets". Sets definable in the enriched language via the satisfaction relation are definable without it, and this reflection provides an absoluteness for definability.

Gödel's [1946] remarks would remain largely unknown in the succeeding two decades. John Myhill and Dana Scott in their [1971] carried out the development of the sets Gödel described, the *ordinal definable* sets. Gödel had at first described the constructible sets informally and shown that being constructible is itself formally definable in ZF; Gödel's claim above entails that being ordinal definable is likewise formally definable in ZF. This Myhill and Scott established with the ZF Reflection Principle, and this speaks to the road not taken by Gödel [1940a] discussed at the end of §3. Moreover, as was anticipated by Gödel [1946, p. 4] the ordinal definable sets provided a new proof for the relative consistency of AC: HOD, the class of *hereditarily ordinal definable* sets is an inner model of ZFC. HOD has become an important feature of modern set theory, and important results about it have articulated Gödel's absolute definability motivation.[37]

[36]See Jané [1995] for the role of the absolute infinite in Cantor's conception of set. Wang [1996, pp. 282ff] reported on how Gödel in 1975 acknowledged Cantor's "Absolute", particularly in connection with a set theory of Ackermann.

[37]Leaping forward, see Steel [1995].

In his article [1947] on Cantor's continuum problem Gödel put emphasis on how his philosophical outlook could be brought to bear on mathematical problems and effect mathematical programs. Of the three possibilities in axiomatic set theory, that CH could be demonstrable, disprovable, or undecidable, Gödel [1947, p. 519] regarded the third as the "most likely", and so advocated the search for a proof of the independence of CH, i.e., to establish Con(ZF) implies Con(ZFC + ¬CH) to complement his own relative consistency result with L. However, Gödel stressed that this would not "settle the question definitively" and turned to the possibility of new axioms. The axioms of set theory do not "form a system closed in itself", and so the "very concept of set on which they are based suggests their extension by new axioms which assert the existence of still further iterations of the operation 'set of'." Gödel then elaborated on the strong axioms of infinity he had alluded to in his [1946] by giving as examples the inaccessible cardinals (as mentioned in §6 in connection with Zermelo [1930]) and the Mahlo cardinals. These were entertained early in the development of set theory and are at the beginning of the modern hierarchy of *large cardinal hypotheses*, hypotheses that posit distinctive structure in the higher reaches of the cumulative hierarchy, most often by positing cardinals whose defining properties entail their inaccessibility from below in strong senses.[38]

Gödel pointed out two significant aspects of large cardinal hypotheses to which attention would be drawn many times in their development. First, in a new twist on the passages quoted in §1, each strong axiom of infinity "can, under the assumption of consistency, be shown to increase the number of decidable propositions even in the field of Diophantine equations." Large cardinal hypotheses establish the *consistency* of ZFC and stronger theories, and so even though they posit distinctive structure high in the cumulative hierarchy they lead to new simple, decidable propositions even about natural numbers.[39] Second, for the inaccessible and Mahlo cardinals and the like, the "undisprovability of the continuum hypothesis ... goes without change". These cardinals relativize to L, i.e., they retain their defining properties in L, and so the existence of these cardinals is consistent with CH.[40]

Gödel went on to speculate about possible strong axioms of infinity based on "hitherto unknown principles", and then, in a well-known passage, argued for new axioms just on extrinsic and pragmatic bases:

> ... even disregarding the intrinsic necessity of some new axiom, and even in case it had no intrinsic necessity at all, a decision

[38]See Kanamori [2003] for the theory of large cardinals.

[39]The specific focus on Diophantine equations could already be seen in a lecture Gödel [193?], which anticipated now well-known work on Hilbert's 10th Problem.

[40]Actually, that inaccessible cardinals relativize to L was already noted in Gödel's first announcement [1938]. It would be a pivotal advance that not all large cardinals relativize to L (see below).

> about its truth is possible also in another way, namely, inductively by studying its 'success', that is, its fruitfulness in consequences and in particular in 'verifiable' consequences, i.e., consequences demonstrable without the new axiom, whose proofs by means of the new axiom, however, are considerably simpler and easier to discover, and make it possible to condense into one proof many different proofs. The axioms for the system of real numbers, rejected by the intuitionists, have in this sense been verified to some extent owing to the fact that analytical number theory frequently allows us to prove number-theoretical theorems which can subsequently be verified by elementary methods. A much higher degree of verification than that, however, is possible. There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline, and furnishing such powerful methods for solving given problems (and even solving them, as far as that is possible, in a constructivistic way) that quite irrespective of their intrinsic necessity they would have to be assumed at least in the same sense as any well established physical theory.

This advocacy of new axioms merely because of their "success" according to "fruitfulness of consequences" interestingly undercuts an avowedly realist position with a pragmatism that dilutes the force of "truth", but is resonant with subsequent investigations. Gödel [1947] concluded by forwarding the remarkable opinion that CH "will turn out to be wrong" since it has as paradoxical consequences the existence of "thin" (in various senses he articulated) sets of reals of the power of the continuum. These examples, one involving one-to-one continuous images, further emphasize how Gödel was aware of and influenced by the articulation of the continuum by the descriptive set theorists (cf. §4).

In 1963 Paul Cohen established the independences Con(ZF) implies Con (ZF + $\neg$AC) and Con(ZF) implies Con(ZFC + $\neg$CH), these being, of course, the inaugural examples of *forcing*, a remarkably general and flexible method for extending models of set theory. If Gödel's construction of L had launched set theory as a distinctive field of mathematics, then Cohen's method of forcing began its transformation into a modern, sophisticated one.

In a published revision [1964] of his [1947] Gödel took into account new developments, most notably Cohen's independence result for CH. As for large cardinals, in a new footnote 20 Gödel cited the emerging work on what are now known as the strongly compact, measurable, weakly compact, and indescribable cardinals, results which in particular showed that these cardinals are far larger in strong senses than the least inaccessible cardinal. Gödel mentioned in particular the pivotal result of Dana S. Scott [1961] that *if there is a measurable cardinal, then* $V \neq L$.[41] In an unpublished, 1966 revision of that footnote Gödel[42] argued that these "extremely strong axioms of infinity of an entirely new kind" are "supported by strong arguments from analogy, e.g., by the fact that they follow from the existence of generalizations of Stone's representation theorem to Boolean algebras with operations on infinitely many elements." He was evidently referring to the compact cardinals. This is the first appearance in his writing of the heuristic of *generalization* for motivating large cardinals. Recalling Cantor's unitary view of the transfinite as seamlessly extending the finite, some properties satisfied by $\aleph_0$ would be too accidental were they not ascribable to higher cardinals in an eternal recurrence.

In the tremendous expansion of set theory following the introduction of forcing, the theory of large cardinals developed a self-fueling momentum of its own and blossomed into a mainstream of set theory far overshadowing Gödel's early speculations. Nowhere would his words be acknowledged as having been a source of inspiration. On the other hand, an articulated and detailed hierarchy of large cardinal hypotheses was developed with the heuristics of reflection and generalization very much in play, and these hypotheses were shown to decide a wide range of strong set-theoretic propositions. Gödel's hopes that large cardinals could settle the continuum problem itself were dispelled by the observation of Levy–Solovay [1967], known by 1964, that small cardinality forcing notions preserve the defining properties of inaccessible large cardinals, so that CH is independent of their postulations. In a 1966 revision of his [1964] Gödel[43] himself implicitly acknowledged this. In a late, unpublished note [1972] Gödel's advocacy of large cardinal hypotheses had two notable modulations. First, he speculated on their possible use to settle, not CH, but questions of "Goldbach type", i.e., Π^0_1 sentences of arithmetic. Second, Gödel pointed to what modern set theorists understand well:

> These principles show that ever more (and ever more complicated) axioms appear during the development of mathematics. For, in order only to understand the axioms of infinity, one must first have developed set theory to a considerable extent.

Extensive work through the 1970s and up to the present day has considerably strengthened the view that the emerging hierarchy of large cardinals

[41] Earlier, in a draft of a (presumably unsent) letter to Tarski of August 1961, Gödel (see Feferman and Dawson, Jr. [2003b, p. 273]) had written: "I have heard it has been proved that there is no two valued denumerably additive measure for the first inacc. number. I still can't believe that this is true, but don't have the time to check it because I am working mainly on phil[osophy]. I understand the proof is based on some work of yours? You probably have heard of Scott's beautiful result that $V \neq L$ follows from the existence of any such measure for any set. I have not checked this proof either but the result does *not* surprise me."

[42] See Feferman [1990, pp. 260ff].

[43] See Feferman [1990, p. 270].

provides *the* hierarchy of exhaustive principles against which all possible consistency strengths can be gauged, a kind of hierarchical completion of ZFC. First, the various hypotheses, though historically contingent, form a *linear* hierarchy with respect to relative consistency strength. Second, a wide range of strong statements arising in set theory and mathematics have been informatively bracketed in consistency strength between two large cardinal hypotheses. The stronger hypothesis implies that there is a forcing extension in which the statement holds; and if the statement holds, there is an L-like inner model satisfying the weaker hypothesis. *Equi*consistency results were established by refining proof ideas and weakening large cardinals to achieve optimal formulations. Throughout, in addition to their "intrinsic" significance, large cardinals amply exhibited "fruitfulness of consequences" by providing the context for quick proofs and illuminating methods, some later found not to require large cardinals at all. These developments have highlighted the contention that large cardinal hypotheses are not a matter of belief, but rather of method. Going far beyond the true and the false, large cardinals have provided the means for understanding strong statements of set theory and mathematics through relative consistency proofs.

Gödel's early advocacy of the search for new axioms can be seen as vindicated by these broad developments, although that vindication has been in much more subtle ways than he could have anticipated. In latter-day accounts, with modern set theory having reached a high degree of sophistication, there have been retrospective analyses that cast Gödel's sparse words across the vast modern landscape of large cardinal hypotheses, crediting them with enunciating "Gödel's program".[44]

Entering his sixties, mostly preoccupied with philosophy and health problems and despite his earlier advocacy of strong axioms of infinity, Gödel would draw on a distant mathematical initiative taken around the time of his birth to address the continuum problem anew.

§**8. Envoi.** In a letter to Cohen of 22 January 1964 Gödel,[45] in connection with possible new uses of forcing, wrote:

> Once the continuum hypothesis is dropped the key problem concerning the structure of the continuum, in my opinion, is what Hausdorff calls the "Pantachie Problem",[1] i.e., the question of whether there exists a set of sequences of integers of power $\aleph_1$ which for any given sequence of integers contains one majoring it from a certain point on. Hausdorff evidently was trying to solve this problem affirmatively (see [Hausdorff [1907]] and [Hausdorff [1909]]). I was always suspecting that, in contrast to the continuum hypothesis, this proposition is correct and perhaps

[44]See Kennedy–van Atten [2004], Koellner [2006], and Hauser [2006] for Gödel's program.

[45]See Feferman and Dawson, Jr. [2003a, pp. 383ff].

even demonstrable from the axioms of set theory. Moreover I have a feeling that, if your method does not yield a proof of independence here, it may lead to a proof of this proposition. At any rate it should be possible to prove the compatibility of the "Pantachie Hypothese" with $2^{\aleph_0} > \aleph_1$.

[1]In German the problem is frequently called "Problem der Wachstumsordnungen". Perhaps there exists some standard English expression for it, too.

In a letter to Stanisław Ulam of 10 February 1964 Gödel,[46] after praising Cohen's work, wrote similarly about the "Pantachie Problem". What Gödel was describing properly has to do with the "Scale Problem" of Hausdorff [1907, p. 152]. ${}^{\omega}\omega$, the set of functions from ω to ω, can be partially ordered according to: $f <^* g$, f is *eventually dominated by* g, *iff* $\exists m \in \omega \forall n \in \omega$ $(m \leq n \rightarrow f(n) < g(n))$. A *$\kappa$-scale* is a subset of ${}^{\omega}\omega$ which according to $<^*$ is cofinal in ${}^{\omega}\omega$ and of ordertype κ. Without further elaboration, we shall extend these concepts in the expected way to other ordered sets besides ω. Hausdorff observed that CH implies that there is an ω_1-scale, and opined that the existence of an ω_1-scale is of significance independently of CH. This is echoed by Gödel in the above passage, but what he was "suspecting" there has an ironic twist.

It soon became known that in Cohen's *original* model for $\neg$CH, i.e., the one resulting from adding many Cohen reals, there is no ω_1-scale. On the other hand, if one adds many (Solovay) random reals to a model of CH, then any ω_1-scale in the ground model remains one in the generic extension.[47] Thus, the existence of ω_1-scales, like CH, comes under the purview of forcing and is independent of ZFC.

Because of its broader involvement in Gödel's later speculations, we review Hausdorff's work on pantachies as such. Most of Hausdorff [1907] is devoted to the analysis of pantachies and the main section V is entitled "On Pantachie Types".[48] The term "pantachie" derives from its initial use by Paul Du Bois–Reymond [1880] to denote everywhere dense subsets of the continuum and then to various notions connected with his work on rates of growth of real-valued functions and on infinitesimals.[49] Hausdorff redefined "pantachie"

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[46]See Feferman and Dawson, Jr. [2003b, p. 298].

[47]Stephen Hechler in his dissertation of 1967 (cf. Hechler [1974]) introduced dominating reals, and by iterating his forcing established the general assertion that if in the sense of the ground model, κ and λ are cardinals of uncountable cofinality such that $2^{\aleph_0} \leq \kappa$ and $\lambda \leq \kappa$, then there is a cardinal-preserving generic extension in which $2^{\aleph_0} = \kappa$ and there is a λ-scale.

[48]See Plotkin [2005] for a penetrating analysis of Hausdorff's work on pantachies and more generally ordered sets, work remarkable for its depth and early appearance.

[49]At the end of his [1880] Du Bois–Reymond maintained that he rather than Cantor had come first to the concept of a dense subset of the continuum. In his book [1882] Du Bois–Reymond explained that his adjective 'pantachish' derives from the Greek words $\pi\alpha\nu\tau\alpha\chi\tilde{\eta}$, $\pi\alpha\nu\tau\alpha\chi o\tilde{\upsilon}$ for "everywhere". For real functions increasing without bound, Du

as a subset of ${}^{\omega}\mathbb{R}$ *maximal* with respect to being linearly ordered by the eventual dominance ordering, and a further refinement led to scales on ${}^{\omega}\omega$. This anticipated Hausdorff's later work on maximal principles, principles equivalent to the Axiom of Choice. For an ordered set $\langle X, < \rangle$, a (κ, λ^*)-*gap* is a set $\{x_\alpha \mid \alpha < \kappa\} \cup \{y_\alpha \mid \alpha < \lambda\} \subseteq X$ such that $x_\alpha < x_\beta < y_\gamma < y_\delta$ for $\alpha < \beta < \kappa$ and $\delta < \gamma < \lambda$, yet there is no $z \in X$ such that $x_\alpha < z < y_\gamma$ for $\alpha < \kappa$ and $\gamma < \lambda$. Pantachies were easily seen to have no countable cofinal or coinitial subset and no (ω, ω^*)-gaps. Regarding pantachies as higher order continua, it was natural to consider whether there could be (ω_1, ω_1^*)-gaps, their absence being a principle of higher-order continuity. Hausdorff established that with CH all pantachies are isomorphic and have (ω_1, ω_1^*)-gaps. In his [1909] he subsequently established that there is a pantachie with an (ω_1, ω_1^*)-gap without appeal to CH, and this recast from ${}^{\omega}\mathbb{R}$ to ${}^{\omega}\omega$ (cf. Hausdorff [1936]) was to become well-known in modern set theory as an "indestructible" ZFC gap, one that cannot be filled with any forcing that preserves $\aleph_1$. Hausdorff [1907, p. 151] asked in the concluding "The Pantachie Problem" subsection whether there could be a pantachie with no (ω_1, ω_1^*)-gaps. Strikingly, Hausdorff [1907, p. 128] had shown earlier that if there were such a pantachie, then $2^{\aleph_0} = 2^{\aleph_1}$ and hence $\neg$CH. This was the first time that a question in ongoing mathematics had entailed the denial of CH.

In the late 1960s Gödel was mostly preoccupied with philosophy; through association with a new generation of set theorists he also kept abreast of the burgeoning developments in the subject. Yet, going his own way and struck by the plausibility of Hausdorff's old formulations, Gödel in 1970 proposed "orders of growth" axioms for deciding the value of $2^{\aleph_0}$ in two handwritten notes [1970a][1970b].[50]

In [1970a], entitled *Some considerations leading to the probable conclusion that the true power of the continuum is* $\aleph_2$, Gödel claimed to establish $2^{\aleph_0} = \aleph_2$ from the following axioms:

(1) For every $n \in \omega$, there is a ω_{n+1}-scale on ${}^{\omega_n}\omega_n$.
(2) In addition, for every $n \in \omega$, the set of all initial segments of all the functions in the ω_{n+1}-scale on ${}^{\omega_n}\omega_n$ has cardinality ω_n.
(3) There is a pantachie with every well-ordered increasing or decreasing descending subset having length at most ω_1.
(4) In addition, the pantachie has no (ω_1, ω_1^*)-gaps.

Bois–Reymond had considered an ordering where $f < g$, $f \sim g$, or $f > g$ according to whether $\lim_{x\to\infty} f(x)/g(x)$ is zero, finite but not zero, or $+\infty$. He had advocated considering those f, g with $f \sim g$ as representing the same "order of infinity" and ranking these orders according to $<$. But of course, there are f, g incomparable according to Du Bois–Reymond's scheme, and on this basis Hausdorff [1907, p. 107] proclaimed that "*the infinitary pantachie* in the sense of Du Bois–Reymond *does not exist*."

[50]See Solovay's introductory note in Feferman [1995, pp. 405ff] and Brendle–Larson–Todorčević [∞] for extensive mathematical analyses.

To modern eyes, there is an affecting, quixotic grandeur to this reaching back to primordial beginnings of set theory to charge the windmill once again. Gödel's only use of (4) was to apply Hausdorff's conclusion that CH fails, and then he argued that (1)–(3) implies that $2^{\aleph_0} \leq \aleph_2$. However, Martin pointed out that the argument does not work, and Solovay (cf. Feferman [1995, pp. 412ff]) elaborated, showing how by adding many random reals it is consistent to have (1)–(3) and the continuum arbitrarily large. On the other hand, Brendle–Larson–Todorčević [∞] showed that there is a substantial part of Gödel's argument that does work to establish $2^{\aleph_0} \leq \aleph_2$ from propositions closely related to (1)–(3).

Gödel [1970a] took his axioms (1) and (2) to entail for all $m < n < \omega$ the existence of ω_{n+1}-scales on ${}^{\omega_n}\omega_m$ such that the set of initial segments of all the functions involved has cardinality ω_n.[51] In his attempted proof, he appealed to such a scale for $n = 2$ and $m = 1$. In fact, the existence of such a scale for $n = 1$ and $m = 0$ already implies CH, and this was the thrust of his [1970b], entitled *A proof of Cantor's continuum hypothesis from a highly plausible axiom about orders of growth*. At its end, Gödel wrote:

> It seems to me this argument gives *much* more likelihood to the truth of Cantor's continuum hypothesis than any counterargument set up to now gave to its falsehood, and it has at any rate the virtue of deriving the power of the set of *all* functions $\omega \longrightarrow \omega$ from that of certain *very* special sets of these functions.

A few years later, in a letter to Abraham Robinson of 20 March 1974 Gödel[52] wrote:

> Hausdorff proved that the existence of a 'continuous' system of orders of growth (i.e., one where every decreasing ω_1-sequence of closed intervals has a non-empty intersection) is incompatible with Cantor's Continuum Hypothesis. Surprisingly the same is true even for a 'dense' system, i.e., one where every decreasing ω_1 sequence of closed intervals, *all of which are larger than some fixed interval I*, has a non[-]empty intersection. I think many mathematicians will consider this to be a strong argument against the Continuum Hypothesis.

Here, the 'continuous' is clear, that there are no (ω_1, ω_1^*)-gaps, but 'dense' is not. Robinson was fatally inflicted with pancreatic cancer and died three weeks after the date of this letter, on 11 April 1974.[53]

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[51] In an unsent letter to Tarski Gödel (see Feferman [1995, p. 424]) soon disavowed this entailment.

[52] See Feferman and Dawson, Jr. [2003b, p. 204].

[53] It is striking to see Gödel offer comfort to a dying colleague by sharing a piece of mathematics with him. Earlier in the letter Gödel had written: "As you know I have unorthodox views about many things. Two of them would apply here: 1. I don't believe that any medical prognosis is 100% certain, 2. The assertion that our ego consists of protein

Wang [1996, p. 89] reported on how Gödel in 1976, two years before his own death, made the following observations:

> The continuum hypothesis may be true, or at least the power of the continuum may be no greater than aleph-two, but the generalized continuum hypothesis is definitely wrong.
>
> I have written up [some material on] the continuum hypothesis and some other propositions. Originally I thought [I had proved] that the power of the continuum is no greater than aleph-two, but there is a lacuna [in the proof]. I still believe the proposition to be true; even the continuum hypothesis may be true.

What are we to make of all this? In his [1947] Gödel had written with authority about the continuum problem, opining that CH would be shown independent, averring that it is actually false particularly because of its implausible consequences for the continuum, and suggesting that new strong axioms of infinity could settle the matter. With the revitalization of set theory after Cohen and perhaps partly spurred by the 1964 Levy–Solovay observation that large cardinal hypotheses have no direct effect on CH, Gödel pursued his rekindled interest in the very old initiatives of Hausdorff and formulated "orders of growth" axioms to inform the continuum problem anew. In this Gödel exhibited a remarkable fluidity, siding with his axioms and letting the mathematics attend to CH, come what may. In the end Gödel's strong mathematical instincts manifested themselves, and with the continuum problem still looming large and despite his "concept of set" and his once-held enthusiasm for large cardinals, he brought in old mathematical ideas from a different quarter and tried to push forward new mathematics. As set theory was to develop after Gödel, there would be a circling back, with deep and penetrating arguments from strong large cardinal hypotheses that, after all, lead to $2^{\aleph_0} = \aleph_2$.[54]

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molecules seems to me one of the most ridiculous every made. I hope you are sharing at least the second opinion with me." Exactly 18 years before to the day, Gödel on 20 March 1956 wrote to his friend von Neumann, dying of bone cancer (see Feferman and Dawson, Jr. [2003b, p. 373]): "I hope and wish that your condition will soon improve further and that the latest achievements of medicine may, if possible, effect a complete cure." Gödel then went on to raise a mathematical issue, giving the first known formulation of the now well-known P = NP problem of computer science (cf. Hartmanis [1989]). There is something quite affecting, almost wry, in Gödel's conviction that mathematics is to trump everything.

[54] See Bekkali [1991], based on of lectures of Todorčević, for the results that the Perfect Forcing Axiom or Stationary Reflection at $\aleph_2$ implies $2^{\aleph_0} = \aleph_2$. See Woodin [1999] for the result that ψ_{AC} implies $2^{\aleph_0} = \aleph_2$.

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