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MODELS OF SECOND-ORDER ZERMELO SET THEORY

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§1. Introduction. In [12], Ernst Zermelo described a succession of models for the axioms of set theory as initial segments of a cumulative hierarchy of levels $\bigcup_\alpha V_\alpha$. The recursive definition of the V_α's is:

$$V_0 = \emptyset; \qquad V_{\alpha+1} = \mathcal{P}(V_\alpha); \qquad V_\lambda = \bigcup_{\beta<\lambda} V_\beta \quad \text{for limit ordinals } \lambda.$$

Thus, a little reflection on the axioms of Zermelo-Fraenkel set theory (ZF) shows that V_ω, the first transfinite level of the hierarchy, is a model of all the axioms of ZF with the exception of the axiom of infinity. And, in general, one finds that if κ is a strongly inaccessible ordinal, then V_κ is a model of all of the axioms of ZF.[1] (For all these models, we take $\in$ to be the standard element-set relation restricted to the members of the domain.) Doubtless, when cast as a first-order theory, ZF does not characterize the structures $\langle V_\kappa, \in \cap (V_\kappa \times V_\kappa)\rangle$ for κ a strongly inaccessible ordinal, by the Löwenheim-Skolem theorem. Still, one of the main achievements of [12] consisted in establishing that a characterization of these models can be attained when one ventures into second-order logic. For let second-order ZF be, as usual, the theory that results from ZF when the axiom schema of replacement is replaced by its second-order universal closure. Then, it is a remarkable result due to Zermelo that second-order ZF can only be satisfied in models of the form $\langle V_\kappa, \in \cap (V_\kappa \times V_\kappa)\rangle$ for κ a strongly inaccessible ordinal.[2]

Now, similarly, if V_λ is an initial segment of the cumulative hierarchy indexed by a limit ordinal $\lambda > \omega$, then $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ is a model of Zermelo set theory, which is the theory whose axioms are all of the

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[1]An ordinal κ is *strongly inaccessible* if and only if $\kappa > \omega$ and κ is regular and a strong limit, that is, if $\lambda < \kappa$, then $2^\lambda < \kappa$.

[2]Cf. [12]. Zermelo's original result is more general. In [12], he allowed for Urelemente, eschewed the axiom of infinity, and used a second-order version of the axiom of foundation to establish that the models of his (second-order) version of ZF $-$ *Inf* are characterized up to isomorphism by two cardinals, the number of their Urelemente and the height of their ordinals. In particular, he characterized the heights of his models as ω or the strongly inaccessible ordinals.

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axioms of ZF with the exception of replacement. One may be tempted to view the axioms of Zermelo set theory as an implicit description of these structures, and, in fact, as adequate to formalize much of mathematical practice, since $\langle V_{\omega+\omega}, \in \cap (V_{\omega+\omega} \times V_{\omega+\omega})\rangle$, the first such model, contains isomorphic copies of the real and complex numbers, subsets and functions on the real numbers, and the rest of the objects studied in classical mathematics. Nevertheless, it is plain, again by the Löwenheim-Skolem theorem, that first-order Zermelo set theory does not characterize the initial segments of the cumulative hierarchy indexed by a limit ordinal $\lambda > \omega$. Still, one may wonder whether a characterization can be attained when the axiom schema of separation of Zermelo set theory is replaced by its second-order universal closure. Are, in particular, the axioms of second-order Zermelo set theory sufficient to characterize the structures $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for λ a limit ordinal greater than ω? If not, what else is required to obtain a theory whose axioms are satisfied in all and only those models isomorphic to some such model?

In this paper, we investigate second-order variants of Zermelo set theory; we are going to show that what are perhaps the most common formulations of second-order Zermelo set theory fail to deliver the existence of a vast array of sets of level ω in the cumulative hierarchy. Moreover, even if the axiom of infinity is refined to deliver the existence of V_ω, which consists of all the hereditarily finite sets, as an immediate consequence, there will still be a variety of models of the Zermelo axioms that are not of the desired form; we shall even establish that there are models of second-order variants of Zermelo set theory in which the element-set relation is not well-founded.

In the last section of the paper, we will turn to the question of what is required to characterize the V_λ's for λ a limit ordinal greater than ω. At a certain point, we shall consider the addition of a new axiom to second-order Zermelo set theory. This axiom explicitly incorporates the cumulative hierarchy view into the theory, and permits us to formulate a second-order variation on Zermelo set theory that characterizes the structures $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for limit ordinals $\lambda > \omega$.

§**2. Axioms of infinity for Zermelo set theory.** There are a variety of alternative formulations of the axiom of infinity, not all of them interderivable. The purpose of this section is to review the relative strength of familiar versions of infinity, and establish the inability of some of these formulations, modulo the rest of the Zermelo axioms (Z^-), to deliver the existence of V_ω, the set of all hereditarily finite sets, as a consequence.

Zermelo's original axiom of infinity asserts the existence of a set which contains the null set and which contains the unit set of every set it contains:

Inf$_Z$ $$\exists y\ (\emptyset \in y \wedge \forall x\ (x \in y \rightarrow \{x\} \in y)).$$

This axiom yields the existence of the set $Z_0 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}$, Zermelo's number sequence, as an immediate consequence, and still occurs in some presentations of standard set theory. $Z^- + \mathit{Inf}_Z$ is the version of Zermelo set theory whose axiom of infinity is Inf_Z.

A more standard formulation of the axiom of infinity is:

Inf $\quad \exists y\, (\emptyset \in y \wedge \forall x\, (x \in y \rightarrow x \cup \{x\} \in y)).$

Inf delivers the existence of ω, the first transfinite (von Neumann) ordinal. According to von Neumann's construction of the ordinals, each ordinal α coincides with the set of its predecessors, $\{\, \beta : \beta < \alpha \,\}$, and $<$ is just the element-set relation on the ordinals. Thus, ω, the first transfinite ordinal, is the set of all finite ordinals, and hence it contains 0 and the successor $\alpha \cup \{\alpha\}$ of every finite ordinal α it contains.[3] *Inf* is what is perhaps the most common version of the axiom of infinity, and I will abbreviate $Z^- + \mathit{Inf}$ as Z, in accordance with the fact that the name Zermelo set theory is most commonly used to refer to $Z^- + \mathit{Inf}$.

The following sentence is an ostensibly weaker axiom of infinity:

InfDed $\quad \exists y\, \exists f\, \exists x\, (\mathrm{Fnc}\, f \wedge x \in y \wedge f : y \rightarrow_{(1\text{-}1)} y - \{x\}).$

Not only does *InfDed* fail to imply either *Inf* or Inf_Z (modulo the axioms of Z^-, of course), as we will see in a moment, it can even be shown that no infinite set is a member of all the models of second-order $Z^- + \mathit{InfDed}$.

InfDed is equivalent, modulo the axioms of Z^-, to the assertion that there exists an ordinary infinite set, a set y which cannot be put in one-one correspondence with any set of natural numbers less than some natural number n. This result is due to Russell who proved that the power set $\mathcal{P}(\mathcal{P}(x))$ of the power set $\mathcal{P}(x)$ of an infinite set x is Dedekind infinite.[4] It should be noted, however, that, absent choice, not only can it not be proved that no infinite set is Dedekind finite, it cannot even be proved that there do not exist infinite sets whose power set is Dedekind finite.[5]

The other, less common formulation of the axiom of infinity I want to consider is:

InfNew $\quad \exists y\, (\emptyset \in y \wedge \forall x\, \forall z\, (x \in y \wedge z \in y \rightarrow x \cup \{z\} \in y)).$

[3]It should perhaps be mentioned that if one's aim were to develop mathematics within Zermelo set theory, then one would probably use a different construction of the ordinals. For, after all, in a model of Zermelo set theory like $V_{\omega+\omega}$, there are no von Neumann ordinals equal to or greater than $\omega + \omega$, and thus no von Neumann ordinals that can be used, for example, to count uncountable well-ordered sets in the domain, $V_{\omega+\omega}$.

[4]If x is infinite, it can be proved that for each natural number n, the set Sn of all subsets of x of cardinality n is nonempty, and if $m \neq n$, Sm and Sn are distinct. But then, $S0$ and the function that assigns $S(n+1)$ to Sn and T itself to each subset T of $\mathcal{P}(x)$ not of the form Sn for some n bears witness to the fact that $\mathcal{P}(\mathcal{P}(x))$ is Dedekind infinite. This result is sometimes erroneously attributed to Tarski, but see [3] for a detailed account.

[5]Cf. [5, ch. 3].

It is evident that this axiom of infinity implies the existence of V_ω, which coincides with *HF*, the set of all hereditarily finite sets, as an immediate consequence, and even though it is mentioned in the second edition of [6] and figures as the official axiom infinity in Azriel Levy's excellent text [7], it is seldom discussed in standard treatments of set theory.

There are, to be sure, other variations on the axiom of infinity in the literature, but I am not now concerned to present an exhaustive review. My aim rather is to point out the existence of important, and often neglected, differences among what are perhaps the most common versions of the axiom of infinity.[6]

The second-order theories $\mathrm{Z}^- + \mathit{InfDed}$, $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$, Z, and $\mathrm{Z}^- + \mathit{InfNew}$ having been set out, the time has come to examine dependencies among them. It is evident that every theorem of second-order $\mathrm{Z}^- + \mathit{InfDed}$, $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$ and Z is a theorem of second-order $\mathrm{Z}^- + \mathit{InfNew}$, but one might inquire whether it is the case that every theorem of second-order $\mathrm{Z}^- + \mathit{InfNew}$ is a theorem of some of the other variations on Zermelo set theory. There is a certain set-theoretic construction that will permit us to answer this question in the negative.[7] If x is a set, define the set $M_n(x)$ by the recursion:

$$M_0(x) = x, \qquad M_{n+1}(x) = M_n(x) \cup \bigcup M_n(x) \cup \mathcal{P}(M_n(x)).$$

Then, the basic closure of x, $M(x)$, is the union

$$M(x) = \bigcup_{n \in \omega} M_n(x).$$

If x is a (pure) transitive set, then it is routine to verify that $M_{n+1}(x)$ is just $\mathcal{P}(M_n(x))$, and that $M(x)$ is a (pure) transitive set which is closed under subsets, and closed under all the Zermelo operations.[8] Thus, $M(\emptyset)$ is V_ω, or, equivalently, *HF*, the set of all hereditarily finite sets, and, in general, $M(x)$ is the domain of the $\subseteq$-least transitive model of Z^- with the standard element-set relation which is closed under subsets and contains the set x. As a consequence, $\langle M(Z_0), \in \cap (M(Z_0) \times M(Z_0))\rangle$ and $\langle M(\omega), \in \cap (M(\omega) \times M(\omega))\rangle$ are, respectively, the $\subseteq$-least transitive models of second-order $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$ and second-order Z with the standard element-set relation.

LEMMA. $M(\omega) \cap M(Z_0) = \mathit{HF}$.

[6] I have suggested that many theorists overlook important differences among common versions of the axiom of infinity, but I would not want to suggest that this oversight is too pervasive. For, as I should emphasize, the relative independence, modulo the axioms of Z^-, of alternative axioms of infinity is mentioned in [1], and in the second edition of [6]. And some of the drawbacks at which we shall look in the course of the discussion have been noticed before in the literature. See for example [4, pp. 110–111] and [9, Appendix B].

[7] The construction appears, for example, in [9, p. 175]. I borrow the term basic closure from Moschovakis.

[8] A set is *closed under subsets* if it contains every subset of each of its members. Proofs of all these basic facts can be found in [9].

PROOF. That $HF \subseteq M(\omega) \cap M(Z_0)$ is an immediate consequence of the fact that both $M(\omega)$ and $M(Z_0)$ contain the null set and are closed under the power set operation.

To verify the converse inclusion, note first that $M_0(Z_0) \cap M_0(\omega) = \{\emptyset, \{\emptyset\}\}$, a member of *HF*. Suppose now that $M_n(Z_0) \cap M_n(\omega)$ is an element of *HF*. Then $M_{n+1}(Z_0) \cap M_{n+1}(\omega) = \mathcal{P}(M_n(Z_0)) \cap \mathcal{P}(M_n(\omega)) = \mathcal{P}(M_n(Z_0) \cap M_n(\omega))$. And since $M_n(Z_0) \cap M_n(\omega) \in HF$, $\mathcal{P}(M_n(Z_0) \cap M_n(\omega)) \in HF$. ⊣

As an immediate consequence of this lemma, we obtain:

THEOREM 1. *There is no $\subseteq$-least transitive model of second-order*

$$\mathrm{Z}^- + \mathit{InfDed}$$

with the standard element-set relation.

Hence we conclude that there is no infinite set whose existence is a logical consequence of second-order $\mathrm{Z}^- + \mathit{InfDed}$. Two other immediate consequences of the lemma are:

THEOREM 2. $\langle M(Z_0), \in \cap (M(Z_0) \times M(Z_0))\rangle$ *is not a model of second-order* Z,

and

THEOREM 3. $\langle M(\omega), \in \cap (M(\omega) \times M(\omega))\rangle$ *is not a model of second-order* $\mathrm{Z}^- + \mathit{Inf}_\mathrm{Z}$.

PROOF OF THEOREMS 2 AND 3. Since $M(Z_0) \cap M(\omega) = HF$ and $Z_0 \notin HF$, $Z_0 \notin M(\omega)$. Likewise, since $\omega \notin HF$, $\omega \notin M(Z_0)$. ⊣

The dependencies established in this section may now be summarized in the following diagram:

$$\begin{array}{ccccc} & \swarrow & \mathrm{Z} + \mathit{InfNew} & \searrow & \\ \mathrm{Z} & & \not\longleftrightarrow & & \mathrm{Z} + \mathit{Inf}_\mathrm{Z} \\ & \searrow & \mathrm{Z}^- + \mathit{InfDed} & \swarrow & \end{array}$$

In this diagram, "$\rightarrow$" abbreviates: "is strictly stronger than," that is, "$\mathrm{Z}^- + \mathit{InfNew} \rightarrow \mathrm{Z}$" says that every theorem of $\mathrm{Z}^- + \mathit{InfNew}$ is a theorem of Z, but that not every theorem of Z is a theorem of $\mathrm{Z}^- + \mathit{InfNew}$. "$\mathrm{Z} \not\leftrightarrow \mathrm{Z}^- + \mathit{Inf}_\mathrm{Z}$" indicates both that there are theorems of Z that are not theorems of $\mathrm{Z}^- + \mathit{Inf}_\mathrm{Z}$ and that there are theorems of $\mathrm{Z}^- + \mathit{Inf}_\mathrm{Z}$ that are not theorems of Z.

One moral to be extracted from these results is that neither of what are perhaps the two most common second-order variants of Zermelo set theory has the resources necessary to guarantee the existence of sets that appear at level ω of the cumulative hierarchy, and are thus quite low down in terms of their cumulative structure—some of these sets are in fact obtainable as the range of a Δ_0 formula with domain ω, and hence minimal in terms of complexity, too.

§3. Well-foundedness, cumulative structure, and the Zermelo axioms. The results of section 2 make plain that, on their more standard formulations, the Zermelo axioms cannot deliver the existence of all sets of level $< \omega + \omega$ in the cumulative hierarchy. One foreseeable reaction to these results is to take them to rest on a common oversight in the standard formulations of the axiom of infinity, since, as we have advanced in section 2, *InfNew* is a variation on the axiom of infinity that implies the existence of V_ω as an immediate consequence.

The interest of second-order $Z^- + \mathit{InfNew}$ is that it is a variant of Zermelo set theory that guarantees the existence of the first $\omega + \omega$ levels of the cumulative hierarchy, and thus the question immediately arises whether this theory is adequate to characterize the initial segments of the cumulative hierarchy that are indexed by a limit ordinal $\lambda > \omega$.

Of course a prerequisite for a (second-order) theory to characterize a class of initial segments of the cumulative structure is that it be satisfiable exclusively in models in which the element-set relation is well-founded. Since the second-order principle of set-theoretic induction is a theorem of second-order ZF, we may rest assured that second-order ZF is a candidate to characterize the initial segments of the cumulative hierarchy that are indexed by some strongly inaccessible ordinal, as in fact it does. But can we, likewise, rest assured that second-order $Z^- + \mathit{InfNew}$ can only be satisfied in models in which the element-set relation is well-founded? We could, if we were in a position to derive the second-order principle of set-theoretic induction as a theorem of second-order $Z^- + \mathit{InfNew}$. Curiously, however, the answer to our question is negative. Not only can the second-order principle of set-theoretic induction not be derived from the axioms of second-order $Z^- + \mathit{InfNew}$, one can even make use of the Rieger-Bernays method[9] for showing the independence of the axiom of foundation to construct models of $Z^- + \mathit{InfNew}$ in which the element-set relation is *not* well-founded.[10]

One may be surprised to hear that that there are non-well-founded models of second-order versions of Zermelo set theory. After all, these theories come equipped with an axiom of regularity or foundation,

$$\mathit{Reg} \qquad \forall x\,(\exists y\,(y \in x) \rightarrow \exists y\,(y \in x \wedge y \cap x = \emptyset)),$$

which is designed precisely to prevent this situation. It is well-known that in the context of first-order set theory the axiom of regularity can only prevent the existence of infinite descending $\in$-chains that are first-order definable in

[9] Cf. [2] and [10].

[10] I am grateful to Vann McGee for asking the question of whether there are non-well-founded models of second-order variants of Zermelo set theory.

the model. But the fault for the failure of the axiom of regularity to prevent the existence of infinite descending $\in$-chains that are not definable in the model is often supposed to lie merely in the fact that the first-order schemata of separation and replacement are ill-suited to capture the full content of these axioms.

Much less well-known is the fact that the axiom of regularity fails, even in the presence of *second-order* separation, to prevent the existence of non-well-founded models of several variants of Zermelo set theory:

THEOREM 4. *There are non-well-founded models of second-order* $\mathrm{Z}^- + \mathit{InfNew}$.

PROOF. To produce a non-well-founded model $\mathcal{M}$ of second-order $\mathrm{Z}^- + \mathit{InfNew}$, take the domain of $\mathcal{M}$ to be $V_{\omega+\omega}$, and let π be a permutation of the domain $V_{\omega+\omega}$ defined by:

$$\begin{aligned} \pi(x) &= \{\{x\}\}, && \text{if } x \in \{Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots\}, \\ \pi(Z_0 - x) &= Z_0 - \bigcup\bigcup x, && \text{if } x \in Z_0 - \{\emptyset, \{\emptyset\}\}, \\ \pi(Z_0 - \{\emptyset\}) &= \{Z_0\}, && \text{and} \\ \pi(x) &= x && \text{otherwise.} \end{aligned}$$

An informal, but more intuitive characterization of π is that it shifts each term forward two steps in the sequence:

$$\dots, Z_0 - \{\{\{\emptyset\}\}\}, Z_0 - \{\{\emptyset\}\}, Z_0 - \{\emptyset\}, Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots .$$

The relation $\in_{\text{new}} \subseteq (V_{\omega+\omega} \times V_{\omega+\omega})$ by which the symbol $\in$ is to be interpreted in $\mathcal{M}$ may then be defined by: $x \in_{\text{new}} y$ if and only if $x \in \pi(y)$. It is then immediate that $\in_{\text{new}}$ is not well-founded in $\mathcal{M}$, as $Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots$ are the members of an an infinite descending $\in_{\text{new}}$-sequence in the model.

We must now see that $\mathcal{M}$ is a model of second-order $\mathrm{Z}^- + \mathit{InfNew}$. It is routine to verify that the truth of the axioms of extensionality, null-set, pairing, infinity, and second-order separation is unaffected by π. The axioms of union, power set, and regularity require more attention and are discussed here:

Union: Let x be a member of $V_{\omega+\omega}$, and note that it is a consequence of our definition of π that

$$\forall x\ (x \in V_{\omega+\omega} \rightarrow \operatorname{rank}(x) \leq \operatorname{rank}(\pi(x)) \leq \operatorname{rank}(x) + 2).[11]$$

[11]As usual, $\operatorname{rank}(x)$, the *rank* of x, is the least ordinal α such that $x \subseteq V_\alpha$. I should emphasize that this feature of π plays an important role in the proof, for, in general, it is not the case that a one-to-one map of $V_{\omega+\omega}$ onto $V_{\omega+\omega}$ induces a model of $\mathrm{Z}^- + \mathit{InfNew}$: A permutation that assigns all the finite Zermelo ordinals to their obvious counterparts in $\{Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots\}$ will generate a model in which union fails—consider the union of Z_0 in such a model. Compare with the Rieger-Bernays method for constructing non-well-founded models of ZF minus regularity.

Since $\{\, \pi(y) : y \in \pi(x) \,\} \subseteq V_{\mathrm{rank}(x)+2}$ and $V_{\mathrm{rank}(x)+2}$ is itself a member of $V_{\omega+\omega}$, $\{\, \pi(y) : y \in \pi(x) \,\} \in V_{\omega+\omega}$. Hence we can infer that

$$\bigcup\{\, \pi(y) : y \in \pi(x) \,\}$$

is a member of $V_{\omega+\omega}$ as well. Now, consider the set

$$\pi^{-1}\Big(\bigcup\{\, \pi(y) : y \in \pi(x) \,\}\Big)$$

and observe that if z is a member of $V_{\omega+\omega}$, then

$$z \in_{\mathrm{new}} \pi^{-1}\Big(\bigcup\{\, \pi(y) : y \in \pi(x) \,\}\Big)$$

just in case $z \in \pi(y)$ for y a member of $V_{\omega+\omega}$ such that $y \in \pi(x)$—just in case $z \in_{\mathrm{new}} y$ for y a member of $V_{\omega+\omega}$ such that $y \in_{\mathrm{new}} x$.

Power: Suppose that x is a member of $V_{\omega+\omega}$, and note that $\pi(x)$ and $\mathcal{P}(\pi(x))$ are members of $V_{\omega+\omega}$. Observe that if $y \in \mathcal{P}(\pi(x))$, then

$$\mathrm{rank}(\pi^{-1}(y)) \leq \mathrm{rank}(y) \leq \mathrm{rank}(\mathcal{P}(\pi(x))).$$

Therefore, since $\{\, \pi^{-1}(y) : y \in \mathcal{P}(\pi(x)) \,\}$ is a member of $V_{\omega+\omega}$,

$$\pi^{-1}(\{\, \pi^{-1}(y) : y \in \mathcal{P}(\pi(x)) \,\})$$

is a member of $V_{\omega+\omega}$ such that if z is a member of $V_{\omega+\omega}$,

$$z \in_{\mathrm{new}} \pi^{-1}(\{\, \pi^{-1}(y) : y \in \mathcal{P}(\pi(x)) \,\})$$

just in case $\pi(z) \subseteq \pi(x)$—just in case

$$\forall w\ (w \in_{\mathrm{new}} z \rightarrow w \in_{\mathrm{new}} x),$$

as desired.

Regularity:

Case 1. Suppose $x \in \{Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots\}$. Then, $\{x\}$ is a $\in_{\mathrm{new}}$-minimal element of x, since $\{x\} \in_{\mathrm{new}} x$, but $\{\{x\}\} \notin_{\mathrm{new}} x$.

Case 2. Suppose $x = Z_0 - y$ with $y \neq \emptyset$. If $y = \{\emptyset\}$, then Z_0 itself is a $\in_{\mathrm{new}}$-minimal member of x. Else, if $y \neq \{\emptyset\}$, then $\emptyset$ is a $\in_{\mathrm{new}}$-minimal member of x.

Case 3. Otherwise, $\forall y \in V_{\omega+\omega}\ (y \in_{\mathrm{new}} x \leftrightarrow y \in x)$. Let y be a $\in$-minimal element of x. If $\pi(y) = y$, then y is a $\in_{\mathrm{new}}$-minimal member of x. Else, if $\pi(y) \neq y$, then we distinguish two subcases: (a) $y \in \{Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots\}$. If $\{y\} \notin x$, then done. Otherwise, let $z \in x \cap \{Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots\}$ such that $\{z\} \notin x$—remember that $x \in V_{\omega+\omega}$, and hence cannot contain all the elements of $\{Z_0, \{Z_0\}, \{\{Z_0\}\}, \dots\}$ as members. Then, z is a $\in_{\mathrm{new}}$-minimal element of x. (b) Else, $y = Z_0 - z$ for some $z \in Z_0$ with $z \neq \emptyset$. If $z = \{\emptyset\}$, then if $Z_0 \in x$, proceed as in Case 1. Otherwise, $\emptyset$ itself is a $\in_{\mathrm{new}}$-minimal member of x. $\dashv$

Thus we conclude that in the absence of replacement, the usual first-order version of the axiom of regularity fails to insure that the axioms of second-order $Z^- + \mathit{InfNew}$ are never satisfied in non-well-founded models.[12] And this result carries over to the second-order variants of Zermelo set theory discussed thus far as well.

Perhaps we should have anticipated this result by reflecting on the fact that the Zermelo axioms are inadequate to prove that every set has a transitive closure,[13] and this is one of the basic facts used in the standard proof that the axiom of regularity implies, in the context of second-order set theory, the second-order principle of set-theoretic induction.

One may well wonder whether amending second-order $Z^- + \mathit{InfNew}$ by adjoining to it an axiom to the effect that every set is contained in some transitive set would suffice to (i) ensure that the set-theoretic universe is well-founded, and (ii) characterize the models of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for λ a limit ordinal greater than ω. These questions are not only different, but require different answers. The answer to question (i) is affirmative, since the second-order principle of set-theoretic induction is a theorem of second-order $Z^- + \mathit{InfNew}$+"Every set has a transitive closure." There is now a well-known construction of models of set theory that will help us settle question (ii) in the negative, since it can be used to produce models of second-order $Z^- + \mathit{InfNew}$+"Every set has a transitive closure" that are not of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for λ a limit ordinal greater than ω: For κ an infinite cardinal, $H(\kappa)$ is the collection of all sets x whose transitive closures contain only sets of cardinality $< \kappa$. It is routine to verify that $H(\kappa)$, for a cardinal $\kappa > \omega$, satisfies all the axioms of (second-order) ZF except possibly power set and replacement. For us, however, the interest of this construction is that it provides us with a recipe to construct models of $Z^- + \mathit{InfNew}$ that are not of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for a limit ordinal $\lambda > \omega$. In particular, if κ is a singular strong limit, then $\langle H(\kappa), \in \cap (H(\kappa) \times H(\kappa))\rangle$ is a model of second-order $Z^- + \mathit{InfNew}$+"Every set has a transitive closure," which is not of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for λ a limit ordinal greater than ω.

Other models of second-order versions of Zermelo set theory that are not of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for a limit ordinal $\lambda > \omega$ can be obtained merely by taking the *basic closure* of a transitive superset of V_ω.[14] Thus for example $M(V_\omega \cup \omega + \omega)$, the basic closure of $V_\omega \cup \omega + \omega$, is the domain of

[12]Replacement is not the only axiom whose absence may distort the content of regularity. It is an old result of Jon Barwise that ZF $-$ *Inf* cannot insure the existence of the transitive closure of a set. Part of the interest of Barwise's result is that it can readily be adapted to show that all the axioms of second-order ZF $-$ *Inf* can be verified in a model in which the extension of the element-set relation is not well-founded.

[13]In [4, pp. 110–111], Frank Drake exhibits a model of (second-order) Z in which not every set has a transitive closure.

[14]As defined in section 2.

another model of second-order Z^-+*InfNew* which contains $\omega + \omega$, but not $V_{\omega+\omega}$ as a member.

§4. Interpreting second-order $Z^- +$ *InfNew* in second-order $Z^- +$ *Inf*$_Z$ and second-order Z. In section 2, we examined dependencies among the second-order theories Z^- + *InfDed*, Z^- + *Inf*$_Z$, Z, and Z^- + *InfNew* to conclude that there is a sense in which Z^- + *InfNew* is undoubtedly superior to the two more standard variants of Zermelo set theory Z^- + *Inf*$_Z$ and Z. Still, there is another question one may raise in investigating the relative strengths of Z^- + *InfNew* and the more familiar Z^- + *Inf*$_Z$ and Z: one may inquire whether they can be interpreted, or at least relatively interpreted in each other. If ϕ is a formula of the language of set theory, let $\phi^{\mathsf{M},\mathsf{E}}$ be the formula that results when $x \in y$ is replaced by the formula $\mathsf{E}(x, y)$ and all quantifiers are relativized to $\mathsf{M}(x)$. As usual, a *relative interpretation* of a version of Zermelo set theory, T_1, in another, T_2, consists of two formulas $\mathsf{M}(x)$ and $\mathsf{E}(x, y)$ which allow one to prove for each axiom ϕ of T_1, the sentence $\phi^{\mathsf{M},\mathsf{E}}$, the interpretation of ϕ, as a theorem of T_2.

Part of the interest of establishing the interpretability of a theory T_1 in another theory T_2 derives from the relative consistency result which immediately follows: If T_1 is interpretable in T_2, then proofs of $\perp$ in T_1 can be turned into proofs of $\perp$ in T_2, and thus the consistency of T_2 implies the consistency of T_1. Yet, the question whether Z^-+*InfNew* can be interpreted in Z^- + *Inf*$_Z$ and Z has an added source of interest. There is no doubt that Z^- + *InfNew* permits the development of a vast part of ordinary mathematics, but, since both Z^- + *Inf*$_Z$ and Z were revealed to be inadequate to secure the existence of a vast array of subsets of V_ω, one may inquire whether they are still adequate to formalize mathematical practice. Establishing the (relative) interpretability of Z^- + *InfNew* in Z^- + *Inf*$_Z$ and Z will show that, for the purposes of formalizing mathematical practice at least, Z^- + *InfNew* is no better than the more standard variants of Zermelo set theory Z^- + *Inf*$_Z$ and Z.

We shall see that Z^- + *InfNew*, Z and Z^- + *Inf*$_Z$ are equi-interpretable, i.e., they all can be (relatively) interpreted in each other. This is of course perfectly compatible with the fact that Z^-+*InfNew* is strictly stronger than both Z and Z^-+*Inf*$_Z$, and can be seen by reflecting on Ackermann's familiar observation that there is a model for ZF minus infinity in the natural numbers: $m \in n$ if and only if the coefficient of 2^m in the binary representation of n is 1.

THEOREM 5. *Z^- + InfNew is relatively interpretable in* Z.

SKETCH OF PROOF. To produce a relative interpretation of Z^- + *InfNew* in Z, define the sequence M_n where $n \in \omega$ by the recursion:

$$M_0 = \omega, \qquad M_{n+1} = \mathcal{P}(M_n) - \mathrm{FIN}(\omega),$$

with $\mathrm{FIN}(\omega) = \{ x \subseteq \omega : x \text{ is finite} \wedge x \notin \omega \}$. The proviso that $x \notin \omega$ is necessary in order to preserve extensionality. Let "$\mathsf{M}(x)$" abbreviate: "$\exists n\ x \in M_n$," and construct a formula "$\mathsf{E}(x, y)$" of the language of Z that expresses the relation $E(x, y)$: "either x and y are members of ω and the binary numeral for y contains a 1 at the 2^x's place, or $x \in y$ otherwise." The trick is to notice that Ackermann's coding can be extended to an isomorphism from $\langle V_{\omega+\omega}, \in \cap (V_{\omega+\omega} \times V_{\omega+\omega})\rangle$ onto $\langle M, E\rangle$. It is then routine to verify that all the interpretations of axioms of $\mathrm{Z}^- + \mathit{InfNew}$ are theorems of Z. $\dashv$

An immediate corollary of this result is that $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$ can be interpreted in Z. And a completely parallel construction establishes both that $\mathrm{Z}^- + \mathit{InfNew}$ can be interpreted in $\mathrm{Z}^- + \mathit{Inf}_{Z}$, and that Z itself can be interpreted in $\mathrm{Z}^- + \mathit{Inf}_{Z}$. Thus it can be concluded that $\mathrm{Z}^- + \mathit{InfNew}$, $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$ and Z are equi-interpretable.[15]

This result provides a comforting response to the question whether Z, or $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$ for that matter, are still sufficient for the development of ordinary mathematics: They still are; $\mathrm{Z}^- + \mathit{InfNew}$, a theory suited to describe an important fragment of the cumulative hierarchy, can be interpreted in both $\mathrm{Z}^- + \mathit{Inf}_{\mathrm{Z}}$ and Z.

§5. Characterizing the V_λ's for limit ordinals $\lambda > \omega$. The results of this paper make plain, I would like to suggest, that what are perhaps the more familiar versions of second-order Zermelo set theory fail to characterize the initial segments of the cumulative hierarchy indexed by a limit ordinal $\lambda > \omega$. The question now remains of what else exactly is required to characterize the V_λ's for limit ordinals $\lambda > \omega$. The purpose of this section is to answer this question by producing a variation of second-order Zermelo set theory whose axioms can only be satisfied in models of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for λ a limit ordinal greater than ω.

The first point to be noticed is that if we take the variables α, β, γ, ... to range over (von Neumann) ordinals, then the V_α's can be characterized thus:

$$\begin{aligned} x = V_\alpha \leftrightarrow \exists f\ (&\mathrm{Fnc}\, f \wedge \mathrm{Dom}(f) = \alpha + 1 \\ &\wedge \forall \beta \leq \alpha\ \forall y\ (y \in f(\beta) \leftrightarrow \exists \lambda < \beta\ (y \subseteq f(\lambda))) \wedge f(\alpha) = x). \end{aligned}$$

And this immediately suggests a formulation of the axiom of regularity which can be used to enforce the modern cumulative view of the set-theoretic universe. This axiom reads:

$$\forall x\ \exists \alpha\ \exists y\ (y = V_\alpha \wedge x \subseteq y).$$

[15] Z, it should be noted, proves relativizations of all the instances of set-theoretic induction and of the axiom asserting the existence of a transitive superset of a set. And perfectly analogous results hold when we restrict our attention to first-order formulations of Zermelo set theory.

Now consider the theory that results from second-order Z when the axiom of regularity, *Reg*, is replaced by the axiom: $\forall x\ \exists\alpha\ \exists y\ (y = V_\alpha \wedge x \subseteq y)$. Then, the distinctions among the axioms of infinity discussed in section 2 collapse, and the axioms of second order $\mathrm{Z} + \forall x\ \exists\alpha\ \exists y\ (y = V_\alpha \wedge x \subseteq y)$ do characterize the V_λ's for limit ordinals $\lambda > \omega$.[16]

Notice first that the second-order principle of set-theoretic induction,

$$\forall X\ (\exists x\ Xx \rightarrow (\exists x\ Xx \wedge \forall y\ (y \in x \rightarrow \neg Xy))),$$

is a theorem of the system. (Suppose Xx. Then, $\exists\alpha\ \exists x\ (Xx \wedge x \subseteq V_\alpha)$, and, by induction on the ordinals,

$$\exists\beta\ (\exists x\ (Xx \wedge x \subseteq V_\beta) \wedge \forall\lambda < \beta\ \neg\exists x\ (Xx \wedge x \subseteq V_\lambda)).$$

Pick such β and x. Then of course, $\forall y \in x\ \neg Xy$, since, otherwise, there would be an ordinal $\lambda < \beta$ such that $Xy \wedge y \subseteq V_\lambda$.)

THEOREM 6. *$\mathcal{M}$ is a model of second-order $\mathrm{Z}+\forall x\ \exists\alpha\ \exists y\ (y = V_\alpha \wedge x \subseteq y)$ if and only if $\mathcal{M}$ is of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for λ a limit ordinal greater than ω.*

SKETCH OF PROOF. Suppose $\mathcal{M}$ is a model of second-order $\mathrm{Z} + \forall x\ \exists\alpha\ \exists y\ (y = V_\alpha \wedge x \subseteq y)$. By the (second-order) principle of set-theoretic induction and extensionality, the $\in$-relation of the model is well-founded and extensional, and, hence, by the Mostowski isomorphism theorem, $\mathcal{M}$ is isomorphic to a transitive $\in$-model. Without loss of generality, let us now confine attention to transitive $\in$-models of $\mathrm{Z}+\forall x\ \exists\alpha\ \exists y\ (y = V_\alpha \wedge x \subseteq y)$. Suppose $\mathcal{M}$ is such a model, and let λ be the least von Neumann ordinal not in the domain. λ is a limit ordinal greater than ω, since the model satisfies the axiom of infinity and is closed under successor. Show that every member of V_λ is a member of the domain. For every $\beta < \lambda$, β is a member of the

[16]Richard Montague devised in [8] a sentence whose models are only initial segments of the cumulative hierarchy V_α indexed by an arbitrary ordinal α. Not much later, in [11], Dana Scott exploited that insight to produce an axiomatization of set theory designed to insure that the set-theoretic universe is arrayed in a cumulative hierarchy of stages or "partial universes." Intuitively, a partial universe is a set of sets of rank less than some one ordinal α, and a partial universe W is *earlier than* a partial universe V if and only if $W \in V$. The axioms of the theory are those of extensionality and separation, an axiom of accumulation according to which the members of a partial universe are the members or subsets of earlier partial universes, and an axiom of restriction according to which every set is a subset of a partial universe. From these axioms, Scott is able to deduce that the partial universes are well-ordered, that the sets are well-founded, and all the usual axioms of set theory other than infinity, replacement, and choice.

Our new axiom here is, in effect, aimed to achieve the same effect achieved by Scott's axiom of restriction: to ensure that every set is a subset of some V_α for some ordinal α. It should come as no surprise that the second-order theory one obtains from adjoining an axiom of infinity to the rest of the axioms of the second-order version of Scott's theory characterizes the V_λ for limit $\lambda > \omega$. The proof of this fact is entirely analogous to the one to be offered for the variation on second-order Zermelo set theory under consideration.

domain, and, since $\forall x\, \exists\alpha\, \exists y\, (y = V_\alpha \wedge x \subseteq y)$, we have that V_β itself is a member of the domain. Therefore, since the domain of $\mathcal{M}$ is transitive, and $V_\lambda = \bigcup\{ V_\beta : \beta < \lambda \}$, we conclude that every member of V_λ is a member of the domain of $\mathcal{M}$. For the converse inclusion, observe that if x is in the model, then $V_{\mathrm{rank}(x)}$ is a subset of the domain. But now, since λ is not in the model, neither are the V_γ's, for $\gamma > \lambda$, subsets of the domain. And thus, given that $V_{\mathrm{rank}(x)}$ is included in the domain for each x in it, we conclude that no set of rank $> \lambda$ is in the model. $\dashv$

§**6. Conclusion.** The axiom of replacement is often regarded a mere closure postulate on the ordinal levels of the cumulative structure with few or no applications within the first $\omega + \omega$ levels of the cumulative hierarchy. The results of the first part of the paper, however, make plain that replacement has important applications at remarkably low levels of the cumulative hierarchy; sometimes, it is even required to ensure that the cumulation of sets described by standard set theory reaches the level omega in the cumulative hierarchy.

Another important effect of adjoining the replacement to the Zermelo axioms is the assurance that the set-theoretic universe is arrayed in a cumulative structure of levels or stages. The results of this paper indicate that, in the absence of replacement, the Zermelo axioms are not adequate to describe the initial segments of the cumulative hierarchy indexed by limit ordinals $\lambda > \omega$. To capture the modern cumulative view of the set-theoretic universe, one could replace the axiom of regularity of second-order Z by the axiom $\forall x\, \exists\alpha\, \exists y\, (y = V_\alpha \wedge x \subseteq y)$, and the result would be an extension of second-order Z that can only be satisfied in models of the form $\langle V_\lambda, \in \cap (V_\lambda \times V_\lambda)\rangle$ for some limit ordinal $\lambda > \omega$.

One foreseeable source of discontent with second-order Z$+\forall x\, \exists\alpha\, \exists y\, (y = V_\alpha \wedge x \subseteq y)$ is that, unlike Zermelo-Fraenkel set theory, this theory enforces the cumulative hierarchy view by brute force, but it is not a natural extension of the Zermelo axioms. Doubtless, some will suggest that the moral to be extracted is that replacement may very well be the only natural principle about sets whose addition to the Zermelo axioms delivers a system of axioms which contains an implicit description of a cumulative hierarchy of levels or stages. What is the force of this consideration, however, is a question I shall not pursue here.[17]

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