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STEP BY RECURSIVE STEP: CHURCH'S ANALYSIS OF EFFECTIVE CALCULABILITY

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> In fact, the only evidence for the freedom from contradiction of *Principia Mathematica* is the empirical evidence arising from the fact that the system has been in use for some time, many of its consequences have been drawn, and no one has found a contradiction. (Church in a letter to Gödel, July 27, 1932)

Abstract. Alonzo Church's mathematical work on computability and undecidability is well-known indeed, and we seem to have an excellent understanding of the context in which it arose. The approach Church took to the underlying conceptual issues, by contrast, is less well understood. Why, for example, was "Church's Thesis" put forward publicly only in April 1935, when it had been formulated already in February/March 1934? Why did Church choose to formulate it then in terms of Gödel's general recursiveness, not his own λ-definability as he had done in 1934? A number of letters were exchanged between Church and Paul Bernays during the period from December 1934 to August 1937; they throw light on critical developments in Princeton during that period and reveal novel aspects of Church's distinctive contribution to the analysis of the informal notion of *effective calculability*. In particular, they allow me to give informed, though still tentative answers to the questions I raised; the character of my answers is reflected by an alternative title for this paper, *Why Church needed Gödel's recursiveness for his Thesis*. In Section 5, I contrast Church's analysis with that of Alan Turing and explore, in the very last section, an analogy with Dedekind's investigation of continuity.

§0. Proem on Church and Gödel. Church's mathematical work on computability and undecidability is well-known, and its development is described, for example, in informative essays by his students Kleene and Rosser. The study of the Church Nachlaß may provide facts for a fuller grasp of this evolution, but it seems that we have an excellent understanding of the context in which the work arose.[1] By contrast, Church's approach to the underlying

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[1] For additional background, cf. Appendix 2 in [58] and Church's letter in Appendix D. It would be of great interest to know more about the earlier interaction with leading logicians and mathematicians: as reported in [24], Church spent part of his time as a National Research Fellow from 1927 to 1929 at Harvard, Göttingen, and Amsterdam.

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conceptual issues is less well understood, even though a careful examination of the published work is already quite revealing. Important material, relevant to both historical and conceptual issues, is contained in the Bernays Nachlaß at the Eidgenössische Technische Hochschule in Zürich. A number of letters were exchanged between Church and Bernays during the period from December 1934 to August 1937; they throw light on critical developments in Princeton[2] and reveal novel aspects of Church's contribution to the analysis of the informal notion of *effective calculability*. That contribution has been recognized by calling the identification of effective calculability with Gödel's general recursiveness, or equivalent notions, *Church's Thesis*.

Church proposed the *definitional* identification publicly for the first time in a talk to the American Mathematical Society on April 19, 1935; the abstract of the talk had been received by the Society already on March 22. Some of the events leading to this proposal (and to the first undecidability results) are depicted in Martin Davis's fascinating paper *Why Gödel did not have Church's Thesis*: Church formulated a version of his thesis via λ-definability in conversations during the period from late 1933 to early 1934;[3] at that time, the reason for proposing the identification was the quasi- empirical fact he expressed also strongly in a letter to Bernays dated January 23, 1935:

> The most important results of Kleene's thesis concern the problem of finding a formula to represent a given intuitively defined function of positive integers (it is required that the formula shall contain no other symbol than λ, variables, and parentheses). The results of Kleene are so general and the possibilities of extending them apparently so unlimited that one is led to the conjecture that a formula can be found to represent any particular constructively defined function of positive integers whatever.

How strongly such quasi-empirical evidence impressed Church is illustrated by the quotation from his letter to Gödel in the motto of my paper; that was written in response to Gödel's question concerning Church's 1932 paper: "If the system is consistent, won't it then be possible to interpret the basic notions in a system of type theory or in the axiom system of set theory, and

[2]This correspondence shows also how closely Bernays followed and interacted with the work of the Princeton group; this is in striking contrast to the view presented in [26]. It should also be noted that Bernays was in Princeton during the academic year 1935–6, resulting in [1]. I assume that Bernays spent only the Fall term, roughly from late September 1935 to around February 1936, in Princeton. In any event, he made the transatlantic voyage, starting from Le Havre on September 20, in the company of Kurt Gödel and Wolfgang Pauli. Due to health reasons, Gödel left Princeton again at the very end of November; cf. [21, pp. 109–110].

[3]Cf. Section 1 and, in particular, Rosser's remarks quoted there.

is there, apart from such an interpretation, any other way at all to make plausible the consistency?"[4]

In the 1935 abstract the thesis was formulated, however, in terms of general recursiveness, and the sole stated reason for the identification is that "other plausible definitions of effective calculability turn out to yield notions that are either equivalent to or weaker than recursiveness". For Davis, this wording "leaves the impression that in the early spring of 1935 Church was not yet certain that λ-definability and Herbrand-Gödel recursiveness were equivalent."[5] Davis's account continues as follows, specifying a particular order in which central results were obtained:

> Meanwhile, Church and Kleene each proved that all λ-definable functions are recursive. Church submitted an abstract of his work on [sic] March 1935, basing himself on recursiveness rather than λ-definability. By the end of June 1935, Kleene had shown that every recursive function is λ-definable, after which Church [10] was able to put his famous work into its final form. Thus while Gödel hung back because of his reluctance to accept the *evidence* for Church's thesis available in 1935 as decisive, Church (who after all was right) was willing to go ahead, and thereby launch the field of recursive function theory. (p. 12)

The accounts in [47] and [56], together with the information provided by Church in his letters to Bernays, make it perfectly clear that the λ-definability of the general recursive functions was known at the very beginning of 1935; it had been established by Rosser and Kleene. The converse was not known when Church wrote his letter of January 23, 1935, but had definitely been established by March. Church wrote on July 15, 1935 his next letter to Bernays and pointed to "a number of developments" that had taken place "in the meantime"; these developments had led to a(n impressive) list of papers, including his own [8] and [10], Kleene's [43] and [44], Rosser's [55], and the joint papers with Kleene, respectively Rosser. Contrary to Davis's "impression", the equivalence was known already in March of 1935 when the abstract was submitted: if the inclusion of λ-definability in recursiveness had not also been known by then, the thesis could not have been formulated coherently in terms of recursiveness.

The actual sequence of events thus differs in striking ways from Davis's account (based on more limited historical information); most importantly, the order in which the inclusions between λ-definability and general recursiveness

[4]The German original: Falls das System widerspruchsfrei ist, wird es dann nicht möglich sein, die Grundbegriffe in einem System mit Typentheorie bzw. im Axiomensystem der Mengenlehre zu interpretieren, und kann man überhaupt auf einem anderen Wege als durch eine solche Interpretation die Widerspruchsfreiheit plausibel machen?

[5]l.c., p. 10.

were established is reversed. This is not just of historical interest, but important for an evaluation of the broader conceptual issues. I claim, and will support through the subsequent considerations, that Church was reluctant to put forward the thesis in writing—until the equivalence of λ-definability and general recursiveness had been established. The fact that the thesis was formulated in terms of recursiveness indicates also that λ-definability was at first, even by Church, not viewed as one among equally natural definitions of effective calculability: the notion just did not arise from an analysis of the intuitive understanding of effective calculability. I conclude that Church was cautious in a similar way as Gödel. Davis sees stark contrasts between the two: in the above quotation, for example, he sees Gödel as "hanging back" and Church as "willing to go ahead"; Gödel is described as reluctant to accept the "evidence for Church's Thesis available in 1935 as decisive". The conversations on which the comparison between Church and Gödel are based took place, however, already in early 1934.[6] Referring to these same conversations, Davis writes (and these remarks immediately precede the above quotation):

> The question of the equivalence of the class of these general recursive functions with the effectively calculable functions was ... explicitly raised by Gödel in conversation with Church. Nevertheless, Gödel was not convinced by the available evidence, and remained unwilling to endorse the equivalence of effective calculability, either with recursiveness or with λ-definability. He insisted ... that it was 'thoroughly unsatisfactory' to *define* the effectively calculable functions to be some particular class without first showing that 'the generally accepted properties' of the notion of effective calculability necessarily lead to this class.

Again, the evidence for the thesis provided by the equivalence, if it is to be taken as such, was not yet available in 1934. Church's and Gödel's developed views actually turn out to be much closer than this early opposition might lead one to suspect. That will be clear, I hope, from the detailed further discussion. The next section reviews the steps towards Church's "conjecture"; then we will look at the equivalence proof and its impact. Church used implicitly and Gödel formulated explicitly an "absoluteness" property for the rigorous concept that is based on an explication of the informal notion as "computation in some logical calculus". Sections 3 and 4 discuss that explication and absoluteness notion. In the next to last section I contrast

[6]In footnote 18 of his [10] Church remarks: "The question of the relationship between effective calculability and recursiveness (which it is here proposed to answer by identifying the two notions) was raised by Gödel in conversation with the author. The corresponding question of the relationship between effective calculability and λ-definability had previously been proposed by the author independently."

their explication with the analysis of Alan Turing and explore, in the final section, an analogy between Turing's analysis and Dedekind's investigation of continuity. Let me mention already here that both Church and Gödel recognized and emphasized the special character of Turing's analysis: Church pointed out that Turing's notion has the advantage (over recursiveness and λ-definability) "of making the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately"; Gödel asserted that Turing's work gives an analysis of the concept of mechanical procedure and that this concept is *shown to be equivalent* with that of a 'Turing machine'.[7]

§1. Effective calculability: a conjecture. The first letter of the extant correspondence between Bernays and Church was mentioned above; it was written by Church on January 23, 1935 and responds to a letter by Bernays from December 24, 1934. Bernays's letter is not preserved in the Zürich Nachlaß, but it is clear from Church's response which issues had been raised in it: one issue concerned the applicability of Gödel's Incompleteness Theorems to Church's systems in the papers [4] and [5], another the broader research program pursued by Church with Kleene and Rosser. Church describes in his letter two "important developments" with respect to the research program. The first development contains Kleene and Rosser's proof, published in their [48], that the set of postulates in [4] and [5] is contradictory. For the second development Church refers to the proof by Rosser and himself that a certain subsystem is free from contradiction. (Cf. Church's letter in Appendix D for a description of the broader context.) The second development is for our purposes particularly significant and includes Kleene's thesis work. Church asserts that the latter provides support for the conjecture that "a formula can be found to represent any particular constructively defined function of positive integers whatever". He continues:

> It is difficult to prove this conjecture, however, or even to state it accurately, because of the difficulty in saying precisely what is meant by "constructively defined". A vague description can be given by saying that a function is constructively defined if a method is given by which its value could be actually calculated for any particular positive integer whatever. Every recursive definition, of no matter how high an order, is constructive, and as far as I know, every constructive definition is recursive.[8]

[7] Gödel's brief and enigmatic remark (as to a proof of the equivalence) is elucidated in [60] and [61].

[8] The quotation continues directly the above quotation from this letter. – Church's paper [6] was given on December 30, 1933 to a meeting of the Mathematical Association; incidentally, Gödel presented his [27] in the very same session of that meeting. Cf. also [31], reviewing [7].

The last remark is actually reminiscent of part of the discussion in [6], where Church claims that " … it appears to be possible that there should be a system of symbolic logic containing a formula to stand for every definable function of positive integers, and I fully believe that such systems exist". (p. 358) From the context it is clear that "constructive definability" is intended, and the latter means minimally that the values of the function can be calculated for any argument. It is equally clear that the whole point of the paper is to propose plausible formal systems that, somehow, don't fall prey to Gödel's Incompleteness Theorems.

> A system of this sort [with levels of different notions of implications, WS] not only escapes our unpleasant theorem that it must be either insufficient or oversufficient, but I believe that it escapes the equally unpleasant theorem of Kurt Gödel to the effect that, in the case of any system of symbolic logic which has a claim to adequacy, it is impossible to prove its freedom from contradiction in the way projected in the Hilbert program. This theorem of Gödel is, in fact, more closely related to the foregoing considerations than appears from what has been said. (p. 360)

Then Church refers to a system of postulates whose consistency can be proved and which probably is adequate for elementary number theory; it seems to be inconceivable to Church that all formal theories should fail to allow the "representation" of the constructively definable functions. Indeed, for the λ-calculus, the positive conjecture had been made by Church in conversation with Rosser tentatively late in 1933, with greater conviction in early 1934. Rosser describes matters in his [56] as follows:

> One time, in late 1933, I was telling him [Church, WS] about my latest function in the LC [Lambda Calculus, WS]. He remarked that perhaps every effectively calculable function from positive integers to positive integers is definable in LC. He did not say it with any firm conviction. Indeed, I had the impression that it had just come into his mind from hearing about my latest function. With the results of Kleene's thesis and the investigations I had been making that fall, I did not see how Church's suggestion could possibly fail to be true. … After Kleene returned to Princeton on February 7, 1934, Church looked more closely at the relation between λ-definability and effective calculability. Soon he decided they were equivalent, … (p. 345)

Kleene put all of these events, except for Church's very first speculations, after his "return to Princeton on February 7, 1934, and before something like the end of March 1934"; see [19, p. 8]. Church discussed these issues also with Gödel who was at that time, early 1934, not convinced by the proposal to identify effective calculability with λ-definability: he called the proposal "thoroughly unsatisfactory".[9] This must have been discouraging to Church, in particular, as Gödel suggested a different direction for supporting such a claim and made later in his lectures a different proposal for a broader notion; Church reports in a letter to Kleene of November 29, 1935:

> His [Gödel's, WS] only idea at the time was that it might be possible, in terms of effective calculability as an undefined notion, to state a set of axioms which would embody the generally accepted properties of this notion, and to do something on that basis. Evidently it occurred to him later that Herbrand's definition of recursiveness, which has no regard to effective calculability, could be modified in the direction of effective calculability, and he made this proposal in his lectures. At that time he did specifically raise the question of the connection between recursiveness in this new sense and effective calculability, but said he did not think that the two ideas could be satisfactorily identified "except heuristically".[10]

This was indeed Gödel's view and was expressed in Note 3 of his 1934 Princeton lectures. The note is attached to the remark that primitive recursive functions have the *important property* that their unique value *can be computed by a finite procedure*—for each set of arguments.

> The converse seems to be true if, besides recursions according to the schema (2) [of primitive recursion; WS], recursions of other forms (e.g., with respect to two variables simultaneously) are admitted. This cannot be proved, since the notion of finite computation is not defined, but it serves as a heuristic principle.

To some it seemed that the note expressed a form of Church's Thesis. However, in a letter of February 15, 1965 to Martin Davis, Gödel emphasized that no formulation of Church's Thesis is implicit in the conjectured equivalence; he explained:

> . . . it is *not true* that footnote 3 is a statement of Church's Thesis. The conjecture stated there only refers to the equivalence of "finite (computation) procedure" and "recursive procedure". However, I was, at the time of these lectures, not at all convinced that my concept of recursion comprises all possible recursions; and in fact the equivalence between my definition and Kleene's . . . is not quite trivial.[11]

In the Postscriptum to his [28] Gödel asserts that the question raised in footnote 3 can now, in 1965, be "answered affirmatively" for his recursiveness

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[9]Church in a letter to Kleene, dated November 29, 1935, and quoted in [19, p. 9]. The conversation took place, according to Davis, "presumably early in 1934"; that is confirmed by Rosser's account on p. 345 of [56].

[10]This is quoted in [19, p. 9], and is clearly in harmony with Gödel's remark quoted below. As to the relation to Herbrand's concept, see the critical discussion in my [58, pp. 83–85]

[11][19, p. 8].

"which is equivalent with general recursiveness as defined today", i.e., with Kleene's μ-recursiveness. I do not understand, how *that* definition could have convinced Gödel that it captures "all possible recursions", unless its use in proofs of Kleene's normal form theorem is also considered. The ease with which "the" normal form theorem allows to establish equivalences between different formulations makes it plausible that *some* stable notion has been isolated; however, the question, whether that notion corresponds to effective calculability has to be answered independently. – The very next section is focused on the equivalence between general recursiveness and λ-definability, but also the dialectical role this mathematical result played for the first published formulation of Church's Thesis.

§2. Two notions: an equivalence proof. In his first letter to Bernays, Church mentions in the discussion of his *conjecture* two precise mathematical results: all primitive recursive, respectively general recursive functions in Gödel's sense can be represented, and that means that they are λ-definable. The first result is attributed to Kleene[12] and the second to Rosser. The letter's remaining three and a half pages (out of a total of six pages) are concerned with an extension of the pure λ-calculus for the development of elementary number theory, consonant with the considerations of [6] described above. The crucial point to note is that the converse of the mathematical result concerning general recursive functions and, thus, the equivalence between λ-definability and general recursiveness is *not* formulated.

Bernays had evidently remarked in his letter of December 24, 1934, that some statements in [5] about the relation of Gödel's theorems to Church's formal systems were not accurate, namely, that the theorems might not be applicable because some very special features of the system of *Principia Mathematica* seemed to be needed in Gödel's proof.[13] Church responds that Bernays's remarks are "just" and then describes Gödel's response to the very same issue:

> Gödel has since shown me, that his argument can be modified in such a way as to make the use of this special property of the system of Principia unnecessary. In a series of lectures here at Princeton last spring he presented this generalized form of his argument, and was able to set down a very general set of conditions such that his theorem would hold of any system of logic which satisfied them.

[12]This fact is formulated also in [42, Part II on p. 223].

[13]Gödel and Church had a brief exchange on this issue already in June and July of 1932. In his letter of July 27, 1932, Church remarks that von Neumann had drawn his attention "last fall" to Gödel's incompleteness paper of 1931 and continues: "I have been unable to see, however, that your conclusions in §4 of this paper apply to my system. Possibly your argument can be modified so as to make it apply to my system, but I have not been able to find a modification of your argument." Cf. Appendix D.

The conditions Church alludes to are found in Section 6 of Gödel's lectures; they include one precise condition that, according to Gödel, *in practice suffices as a substitute for the unprecise* requirement that the class of axioms and the relation of immediate consequence be constructive. The unprecise requirement is formulated at the beginning of Gödel's lectures to characterize crucial normative features for a *formal mathematical system*:

> We require that the rules of inference, and the definitions of meaningful formulas and axioms, be constructive; that is, for each rule of inference there shall be a finite procedure for determining whether a given formula B is an immediate consequence (by that rule) of given formulas $A_1, \ldots, A_n$, and there shall be a finite procedure for determining whether a given formula A is a meaningful formula or an axiom. (p. 346)

The precise condition replaces "constructive" by "primitive recursive".[14] Not every constructive function is primitive recursive, however: Gödel gives in Section 9 a function of the Ackermann type, asks what one might mean "by every recursive function", and defines in response the class of *general recursive functions* via his equational calculus.

Clearly, it is of interest to understand, why Church *publicly* announced the *thesis* only in his talk of April 19, 1935, and why he formulated it then in terms of general recursiveness, not λ-definability. Here is the full abstract of Church's talk:

> Following a suggestion of Herbrand, but modifying it in an important respect, Gödel has proposed (in a set of lectures at Princeton, N.J., 1934) a definition of the term *recursive function*, in a very general sense. In this paper a definition of *recursive function of positive integers* which is essentially Gödel's is adopted. And it is maintained that the notion of an effectively calculable function of positive integers should be identified with that of a recursive function, since other plausible definitions of effective calculability turn out to yield notions that are either equivalent to or weaker than recursiveness. There are many problems of elementary number theory in which it is required to find an effectively calculable function of positive integers satisfying certain conditions, as well as a large number of problems in other fields which are known to be reducible to problems in number theory of this type. A problem of this class is the problem to find a complete set of invariants of formulas under the operation of conversion (see abstract 41.5.204). It is proved that this problem is unsolvable, in the sense that there is no complete set of effectively calculable invariants.[15]

[14]Here and below I use "primitive recursive" where Gödel just says "recursive" to make explicit the terminological shift that has taken place since (Gödel 1934).

[15][7]. In the next to last sentence "abstract 41.5.204" refers to [16].

Church's letter of July 15, 1935, to Bernays explicitly refers to this abstract and mentions the paper [10] as "in the process of being typewritten"; indeed, Church continues " ... I will mail you a copy within a week or two. All these papers will eventually be published, but it may be a year or more before they appear." His mailing included a copy of a joint paper with Rosser, presumably their [17], and an abstract of a joint paper with Kleene, presumably their [14]. Of historical interest is furthermore that Kleene's papers *General recursive functions of natural numbers* and *λ-definability and recursiveness* are characterized as "forthcoming", i.e., they had been completed already at this time.

The precise connection between recursiveness and λ-definability or, as Church puts it in his abstract, "other plausible definitions of effective calculability" had been discovered in 1935, between the writing of the letters of January 23 and July 15. From the accounts in [47] and [56] it is quite clear that Church, Kleene, and Rosser contributed to the proof of the equivalence of these notions. Notes 3, 16, and 17 in [10] add detail: consistently with the report in the letter to Bernays, the result that all general recursive functions are λ-definable was first found by Rosser and then by Kleene (for a slightly modified definition of λ-definability); the converse claim was established "independently" by Church and Kleene "at about the same time". However, neither from Kleene's or Rosser's historical accounts nor from Church's remarks is it clear, *when* the equivalence was actually established. In view of the letter to Bernays and the submission date for the abstract, March 22, 1935, the proof of the converse must have been found after January 23, 1935, but before March 22, 1935. So one can assume with good reason that this result provided to Church the additional bit of evidence for actually publishing the thesis.[16]

That the thesis was formulated for general recursiveness is not surprising when Rosser's remark in his [56] about this period is *seriously* taken into account: "Church, Kleene, and I each thought that general recursivity seemed to embody the idea of effective calculability, and so each wished to show it equivalent to λ-definability". (p. 345) There was no independent motivation for λ-definability to serve as a concept to capture effective calculability, as the historical record seems to show: consider the surprise that the predecessor function is actually λ-definable and the continued work in 1933/4 by Kleene and Rosser to establish the λ-definability of more and more constructive functions. In addition, Church argued for the correctness of the thesis when completing the 1936 paper (before July 15, 1935); his argument took the form of an *explication* of effective calculability with a central appeal to "recursivity". Kleene referred to Church's analysis, when presenting his

[16]This account should be compared with the more speculative one given in [19], for example in the summary on p. 13.

[45] to the American Mathematical Society on January 1, 1936, and made these introductory remarks (on p. 544): "The notion of a recursive function, which is familiar in the special cases associated with primitive recursions, Ackermann-Péter multiple recursions, and others, has received a general formulation from Herbrand and Gödel. The resulting notion is of especial interest, since the intuitive notion of a 'constructive' or 'effectively calculable' function of natural numbers can be identified with it very satisfactorily." λ-definability was not even mentioned.

§**3. Reckonable functions: an explication.** The paper *An unsolvable problem of elementary number theory* was published, as Church had expected, in (early) 1936. Church restates in it his proposal for identifying the class of effectively calculable functions with a precisely defined class, so that he can give a rigorous mathematical definition of the class of number theoretic problems of the form: "Find an effectively calculable function that is the characteristic function of a number theoretic property or relation." This and an additional crucial point are described by Church as follows:

> The purpose of the present paper is to propose a definition of effective calculability which is thought to correspond satisfactorily to the somewhat vague intuitive notion in terms of which problems of this class are often stated, and to show, by means of an example, that not every problem of this class is solvable.[17]

In Section 7 of his paper, Church presents arguments in support of the proposal to use general recursiveness[18] as the precise notion; indeed, the arguments are to justify the identification "so far as positive justification can ever be obtained for the selection of a formal definition to correspond to an intuitive notion".[19] Two methods to characterize effective calculability of number-theoretic functions suggest themselves. The first of these methods makes use of the notion of *algorithm*, and the second employs the notion of *calculability in a logic*. Church argues that neither method leads to a definition more general than recursiveness. The two arguments have a very similar structure, and I will discuss only the one pertaining to the second method.[20] Church considers a logic **L**, whose language contains the equality symbol $=$, a symbol $\{\}()$ for the application of a unary function symbol to

[17] [10] in [18, pp. 89 and 90].

[18] The fact that λ-definability is an equivalent concept adds for Church " . . . to the strength of the reasons adduced below for believing that they [these precise concepts] constitute as general a characterization of this notion (i.e., effective calculability) as is consistent with the usual intuitive understanding of it". [10], footnote 3, p. 90 in [18].

[19] [10], in [18, p. 100].

[20] An argument following quite closely Church's considerations pertaining to the first method is given in [57, p. 120]. – For the second argument, Church uses the fact that Gödel's class of general recursive functions is closed under the μ-operator, then still called Kleene's

an argument, and numerals for the positive integers. For unary functions F he defines:

> F is *effectively calculable* if and only if there is an expression f in the logic **L** such that: $\{\mathrm{f}\}(\mu) = \nu$ is a theorem of **L** iff $\mathrm{F}(m) = n$; here, μ and ν are expressions that stand for the positive integers m and n.[21]

Such functions F are recursive, if it is *assumed* that **L** satisfies conditions that make **L**'s theorem predicate recursively enumerable. To argue then for the recursive enumerability of the theorem predicate, Church formulates conditions *any* system of logic has to satisfy if it is "to serve at all the purposes for which a system of symbolic logic is usually intended".[22] These conditions, Church remarks in footnote 20, are "substantially" those from Gödel's Princeton Lectures for a formal mathematical system: (**i**) each rule must be an effectively calculable operation, and (**ii**) the set of rules and axioms (if infinite) must be effectively enumerable. Church supposes that these conditions can be *interpreted* to mean that, via a suitable Gödel numbering for the expressions of the logic, ($\mathbf{i}_c$) each rule must be a recursive operation, ($\mathbf{ii}_c$) the set of rules and axioms (if infinite) must be recursively enumerable, and ($\mathbf{iii}_c$) the relation between a positive integer and the expression which stands for it must be recursive. The theorem predicate is thus indeed recursively enumerable, but the *crucial interpretative step* is not argued for at all!

Church's argument in support of the recursiveness of effectively calculable functions may appear to be *viciously circular*. However, our understanding of the general concept of calculability is explicated in terms of derivability in a logic, and the conditions ($\mathbf{i}_c$)–($\mathbf{iii}_c$) sharpen the idea that within such a logical formalism one operates with an effective notion of immediate consequence.[23] The "thesis" is appealed to in a special and narrower context, and it is precisely here that we encounter the real stumbling block for Church's analysis. Given the crucial role this observation plays, it is appropriate to formulate it as a normative requirement:

CHURCH'S CENTRAL THESIS. The steps of any effective procedure (governing proofs in a system of symbolic logic) must be recursive.

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p-function. That result is not needed for the first argument on account of the determinacy of algorithms.

[21]This concept is an extremely natural and fruitful one and is directly related to "Entscheidungsdefinitheit" for relations and classes introduced by Gödel in his 1931 paper and to representability of functions used in his 1934 Princeton Lectures. As to the former, compare *Collected Works I*, pp. 170 and 176; as to the latter, see p. 58 in [18].

[22][10] in [18], p. 101. As to what is intended, namely for **L** to satisfy epistemologically motivated restrictions, see [13, Section 07], in particular pp. 52–53.

[23]Compare footnote 20 on p. 101 in [18] where Church remarks: "In any case where the relation of immediate consequence is recursive it is possible to find a set of rules of procedure, equivalent to the original ones, such that each rule is a (one-valued) recursive operation, and the complete set of rules is recursively enumerable."

If the central thesis is accepted, the earlier considerations indeed prove that all effectively calculable functions are recursive. Robin Gandy called this Church's "step-by-step argument".[24]

The idea that computations are carried out in a logic or simply in a deductive formalism is also the starting point of the considerations in a supplement to Hilbert and Bernays's book *Grundlagen der Mathematik II*. Indeed, Bernays's letter of December 24, 1938 begins with an apology for not having written to Church in a long time:

> I was again so much occupied by the working at the "Grundlagenbuch". In particular the "Supplemente" that I found desirable to add have become much more extended than expected. By the way: one of them is on the precising [sic!] of the concept of computability. There I had the opportunity of exposing some of the reasonings of yours and Kleene on general recursive functions and the unsolvability of the Entscheidungsproblem.

Bernays refers to the book's Supplement II, entitled "Eine Präzisierung des Begriffs der berechenbaren Funktion und der Satz von CHURCH über das Entscheidungsproblem". A translation of the title, not quite adequate to capture "Präzisierung", is "A precise explication of the concept of calculable function and Church's Theorem on the decision problem".

In this supplement Hilbert and Bernays make the core notion of *calculability in a logic* directly explicit and define a number theoretic function to be *reckonable* (in German, *regelrecht auswertbar*) when it is computable in some deductive formalism satisfying three recursiveness conditions. The crucial condition is an analogue of Church's Central Thesis and requires that the theorems of the formalism can be enumerated by a primitive recursive function or, equivalently, that the proof predicate is primitive recursive. Then it is shown (1) that a special, very restricted number theoretic formalism suffices to compute the reckonable functions, and (2) that the functions computable in this formalism are exactly the general recursive ones. The analysis provides, in my view, a natural and most satisfactory capping of the development from *Entscheidungsdefinitheit* of relations in Gödel's incompleteness paper of

[24]It is most natural and general to take the underlying generating procedures directly as finitary inductive definitions. That is Post's approach via his production systems; using Church's central thesis to fix the restricted character of the generating steps guarantees the recursive enumerability of the generated set. Cf. Kleene's discussion of Church's argument in [46, pp. 322–323]. Here it might also be good to recall remarks of C. I. Lewis on "inference" as reported in [20] on page 273: "The main thing to be noted about this operation is that it is not so much a piece of reasoning as a mechanical, or strictly mathematical, operation for which a rule has been given. No "mental" operation is involved except that required to recognize a previous proposition followed by the main implication sign, and to set off what follows that sign as a new assertion."

1931 to an "absolute" notion of computability for functions, because it captures directly the informal notion of rule-governed evaluation of effectively calculable functions and isolates appropriate restrictive conditions.

§4. Absoluteness and formalizability. A technical result of the sort I just discussed was for Gödel in 1935 the first hint that there might be a precise notion capturing the informal concept of effective calculability.[25] Gödel defined an absoluteness notion for the specific formal systems of his paper [30]. A number theoretic function $\phi(x)$ is said to be *computable in S* just in case for each numeral m there exists a numeral n such that $\phi(m) = n$ is provable in S. Clearly, all primitive recursively defined functions, for example, are already computable in the system S_1 of classical arithmetic, where S_i is number theory, of order i for i finite or transfinite. In the Postscriptum to the paper Gödel observed:

> It can, moreover, be shown that a function computable in one of the systems S_i, or even in a system of transfinite order, is computable already in S_1. Thus the notion 'computable' is in a certain sense 'absolute', while almost all metamathematical notions otherwise known (for example, provable, definable, and so on) quite essentially depend upon the system adopted. (p. 399)

A broader notion of *absoluteness* was used in Gödel's contribution to the Princeton bicentennial conference, i.e., in [32]. Gödel starts out with the following remark:

> Tarski has stressed in his lecture (and I think justly) the great importance of the concept of general recursiveness (or Turing's computability). It seems to me that this importance is largely due to the fact that with this concept one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending on the formalism chosen. (p. 150)

For the publication of the paper in [18] Gödel added a footnote to the last sentence:

> To be more precise: a function of integers is computable in any formal system containing arithmetic if and only if it is computable in arithmetic, where a function f is called computable in S if there is in S a computable term representing f. (p. 150)

[25]Cf. my [58, p. 88 and Note 52]. The latter asserts that the content of [30] was presented in a talk in Vienna on June 19, 1935. An interesting question is, certainly, how much Gödel knew then about the ongoing work in Princeton reported in Church's 1935 letters to Bernays. I could not find any evidence that Gödel communicated with Bernays, Church, or Kleene on these issues at that time.

Both in 1936 and in 1946, Gödel took for granted the formal character of the systems and, thus, the elementary character of their inference or calculation steps. Gödel's claim that "an absolute definition of an interesting epistemological notion" has been given, i.e., a definition that does not depend on the formalism chosen, is only partially correct: the definition does not depend on the details of the formalism, but depends crucially on the fact that we are dealing with a "formalism" in the first place. In that sense absoluteness has been achieved only relative to an un-explicated notion of an elementary formalism. It is in this conceptual context that Church's letter from June 8, 1937 to the Polish logician Józef Pepis should be seen.[26] Church brings out this "relativism" very clearly in an indirect way of defending his thesis; as far as I know, this broadened perspective, though clearly related to his earlier explication, has not been presented in any of Church's writing on the subject.

Pepis had described to Church his project of constructing a number theoretic function that is effectively calculable, but not general recursive. In his response Church explains, why he is "extremely skeptical". There is a minimal condition for a function f to be effectively calculable, and "if we are not agreed on this then our ideas of effective calculability are so different as to leave no common ground for discussion": for every positive integer a there must exist a positive integer b such that the proposition $f(a) = b$ has a "valid proof" in mathematics. But as all of extant mathematics is formalizable in Principia Mathematica or in one of its known extensions, there actually must be a *formal proof* of a suitably chosen formal proposition. However, if f is not general recursive then, by the considerations of [10], for every definition of f within the language of Principia Mathematica there exists a positive integer a such that for no b the proposition $f(a) = b$ is provable in Principia Mathematica; that holds again for all known extensions. Indeed, Church claims this holds for "any system of symbolic logic whatsoever which to my knowledge has ever been proposed". Thus, to satisfy the above minimal condition and to respect the quasi-empirical fact that all of mathematics is formalizable, one would have to find "an utterly new principle of logic, not only never before formulated, but also never before actually used in a mathematical proof".

Moreover, and here is the indirect appeal to the recursivity of steps, the new principle "must be of so strange, and presumably complicated, a kind that its metamathematical expression as a rule of inference was not general recursive", and one would have to scrutinize the "alleged effective applicability of the principle with considerable care". The dispute concerning a proposed effectively calculable, non-recursive function would thus for Church

[26]The letter to Pepis was partially reproduced and analyzed in my [58]; it is reprinted in Appendix A.

center around the required new principle and its effective applicability as a rule of inference, i.e., what I called Church's Central Thesis. If the latter is taken for granted (implicitly, for example, in Gödel's absoluteness considerations), then the above minimal understanding of effective calculability and the quasi-empirical fact of formalizability block the construction of such a function. This is not a completely convincing argument; Church is extremely skeptical of Pepis's project, but mentions that "this [skeptical] attitude is of course subject to the reservation that I may be induced to change my opinion after seeing your work".

On April 22, 1937, Bernays wrote a letter to Church and remarked that Turing had just sent him the paper [63]; there is a detailed discussion of some points concerned with Turing's proof of the undecidability of the Entscheidungsproblem. As to the general impact of Turing's paper Bernays writes:

> He [Turing] seems to be very talented. His concept of computability is very suggestive and his proof of the equivalence of this notion with your λ-definability gives a stronger conviction of the adequacy of these concepts for expressing the popular meaning of "effective calculability".

Bernays does not give in this letter (or in subsequent letters to Church and to Turing) a reason, why he finds Turing's concept "suggestive"; strangely enough, in Supplement II of *Grundlagen der Mathematik II*, Turing's work is not even mentioned. It is to that work that I'll turn now to indicate in what way it overcomes the limitations of the earlier analyses (all centering around the concept of "computability in a formal logic").

§5. Computors, boundedness, and locality. The earlier detailed reconstruction of Church's justification for the "selection of a formal definition to correspond to an intuitive notion" and the pinpointing of the crucial difficulty show, first of all, the sophistication of Church's methodological attitude and, secondly, that at this point in 1935 there is no major opposition to Gödel's cautious attitude. These points are supported by the directness with which Church recognized in 1937, when writing a review of [63] for the Journal of Symbolic Logic, the importance of Turing's work as making the identification of effectiveness and (Turing) computability "immediately evident". That review is quoted now in full:

> The author proposes as criterion that an infinite sequence of digits 0 and 1 be "computable" that it is possible to devise a computing machine, occupying a finite space and with working parts of finite size, which will write down a sequence to any desired number of terms if allowed to run for a sufficiently long time. As a matter of convenience, certain further restrictions are imposed on the character

> of the machine, but these are of such a nature as obviously to cause no loss of generality - in particular, a human calculator, provided with pencil and paper and explicit instructions, can be regarded as a kind of Turing machine. It is thus immediately clear that computability, so defined, can be identified with (especially, is no less general than) the notion of effectiveness as it appears in certain mathematical problems (various forms of the Entscheidungsproblem, various problems to find complete sets of invariants in topology, group theory, etc., and in general any problem which concerns the discovery of an algorithm).
>
> The principal result is that there exist sequences (well-defined on classical grounds) which are not computable. In particular the *deducibility problem* of the functional calculus of first order (Hilbert and Ackermann's engere Funktionenkalkül) is unsolvable in the sense that, if the formulas of this calculus are enumerated in a straightforward manner, the sequence whose *n*th term is 0 or 1, according as the *n*th formula in the enumeration is or is not deducible, is not computable. (The proof here requires some correction in matters of detail.)
>
> In an appendix the author sketches a proof of equivalence of "computability" in his sense and "effective calculability" in the sense of the present reviewer (***American Journal of Mathematics***, vol. 58 (1936), pp. 345–363, see review in this journal, vol. 1, pp. 73–74). The author's result concerning the existence of uncomputable sequences was also anticipated, in terms of effective calculability, in the cited paper. His work was, however, done independently, being nearly complete and known in substance to a number of people at the time that the paper appeared.
>
> As a matter of fact, there is involved here the equivalence of three different notions: computability by a Turing machine, general recursiveness in the sense of Herbrand-Gödel-Kleene, and the λ-definability in the sense of Kleene and the present reviewer. Of these, the first has the advantage of making the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately - i.e. without the necessity of proving preliminary theorems. The second and third have the advantage of suitability for embodiment in a system of symbolic logic.

So, Turing's notion is presumed to make the identification with effectiveness in the ordinary sense "evident immediately". How this is to be understood is a little clearer from the first paragraph of the review, where it is claimed to be immediately clear "that computability, so defined, can be identified with … the notion of effectiveness as it appears in certain mathematical problems … ". This claim is connected to previous sentences by "thus": the premises of this "inference" are (1) computability is defined via computing machines (that occupy a finite space and have working parts of finite size), and (2) human calculators, "provided with pencil and paper and explicit instructions", can be regarded as Turing machines.

The review of Turing's paper is immediately followed by Church's review of [54]; the latter is reprinted in Appendix C. Church is sharply critical of Post; this is surprising, perhaps, as Church notices the equivalence of Post's and Turing's notions. The reason for the criticism is methodological: Post does not "identify" his formulation of a finite 1-process with effectiveness in the ordinary sense, but rather considers as a "working hypothesis" that wider and wider formulations can be reduced to this formulation; he believes that the working hypothesis is in need of continual verification. Church objects "that effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning". The need for a working hypothesis disappears, so Church argues, if effectiveness is defined as "computability by an arbitrary machine, subject to restrictions of finiteness". The question here is, why does that seem "to be an adequate representation of the ordinary notion"? Referring back to the "inference" isolated in the review of Turing's paper, we may ask, why do the two premises support the identification of Turing computability with the informal notion of effectiveness as used for example in the formulation of the decision problem? Thus we are driven to ask the more general question, what is the real character of Turing's analysis?[27]

Let me emphasize that Turing's analysis is neither concerned with *machine computations* nor with general *human mental processes*. Rather, it is *human mechanical computability* that is being analyzed, and the special character of this intended notion motivates the restrictive conditions that are brought to bear by Turing.[28] Turing exploits in a radical way that a *human computor* is performing *mechanical procedures* on *symbolic configurations*: the immediate recognizability of symbolic configurations is demanded so that basic (computation) steps cannot be further subdivided. This demand and the evident limitation of the computor's sensory apparatus lead to the formulation of boundedness and locality conditions. Turing requires also a *determinacy condition* (**D**), i.e., the computor carries out deterministic computations, as his internal state together with the observed configuration fixes uniquely the next computation step. The *boundedness conditions* can be formulated as follows:

(**B.1**) *there is a fixed bound for the number of symbolic configurations a computor can immediately recognize*;

[27]The following analysis was given in my [58]; it is also presented in the synoptic [61].
[28]This is detailed in my [58].

(**B.2**) *there is a fixed bound for the number of a computor's internal states that need to be taken into account.*[29]

For a given computor there are consequently only boundedly many different combinations of symbolic configurations and internal states. Since his behavior is, according to (**D**), uniquely determined by such combinations and associated operations, the computor can carry out at most finitely many different operations. These operations are restricted by the following *locality conditions*:

(**L.1**) *only elements of observed configurations can be changed*;

(**L.2**) *the distribution of observed squares can be changed, but each of the new observed squares must be within a bounded distance of an immediately previously observed square.*[30]

Turing's computor proceeds deterministically, must satisfy the boundedness conditions, and the elementary operations he can carry out must be restricted as the locality conditions require. Every number-theoretic function such a computor can calculate, Turing argues, is actually computable by a Turing machine over a two-letter alphabet. Thus, on closer inspection, Turing's Thesis that the concept "mechanical procedure" can be identified with machine computability is seen as the result of a two part analysis. The first part yields axioms expressing boundedness conditions for symbolic configurations and locality conditions for mechanical operations on them, together with the *central thesis* that any mechanical procedure can be carried out by a computor satisfying the axioms. The second part argues for the *claim* that every number-theoretic function calculable by such a computor is computable by a Turing machine. In Turing's presentation these quite distinct aspects are intertwined and important steps in arguments are only hinted at.[31] Indeed, the claim that is actually established in Turing's paper is the more modest one that Turing machines operating on strings can be simulated by Turing machines operating on single letters.

In the historical context in which Turing found himself, he asked exactly the right question: What are the elementary processes a computor carries out (when calculating a number)? Turing was concerned with *symbolic processes*, not—as the other proposed explications—with processes directly related to the evaluation of (number theoretic) functions. Indeed, the general "problematic" required an analysis of the idealized capabilities of a computor, and

[29] This condition (and the reference to internal states) can actually be removed and was removed by Turing; nevertheless, it has been a focus of critical attention.

[30] This is almost literally Turing's formulation. Obviously, it takes for granted particular features of the precise model of computation, namely, to express that the computor's attention can be shifted only to symbolic configurations that are not "too far away" from the currently observed configuration.

[31] Turing's considerations are sharpened and generalized in [60].

it is precisely this feature that makes the analysis epistemologically significant. The separation of conceptual analysis and rigorous proof is essential for clarifying on what the correctness of Turing's central thesis rests, namely, on recognizing that the boundedness and locality conditions are true for a computor and also for the particular precise, analyzed notion.

§6. **Conceptual analyses: a brief comparison.** Church's way of approaching the problem was at first deeply affected by quasi-empirical considerations. That is true also for his attitude to the consistency problem for the systems in [4] and [5]; his letter of July 27, 1932 to Gödel is revealing. His review of Turing's 1936 paper shows, however, that he moved away from that position; how far is perhaps even better indicated by the very critical review of [54]. In any event, Turing's approach provides immediately a detailed conceptual analysis realizing, it seems, what Gödel had suggested in conversation with Church, namely "to state a set of axioms which would embody the generally accepted properties of this notion [effective calculability, WS], and to do something on that basis". The analysis leads convincingly to the conclusion that "effectively calculable" functions can be computed by Turing machines (over a two letter alphabet). The latter mathematical notion, appropriately, serves as the starting point for *Computability Theory*; cf. [62].

Turing's analysis divides, as I argued in the last section, into conceptual analysis and rigorous proof. The conceptual analysis leads *first* to a careful and sharper formulation of the intended informal concept, here, "mechanical procedures carried out by a human computor", and *second* to the axiomatic formulation of determinacy, boundedness, and locality conditions. Turing's central thesis connects the informal notion and the axiomatically restricted one. Rigorous proof allows us then, *third*, to recognize that all the actions of an axiomatically restricted computor can be simulated by a Turing machine. Thus, the analysis together with the proof allows us to "replace" the boldly claimed thesis, all effectively calculable functions are Turing computable, by a carefully articulated argument that includes a sharpened informal notion and an axiomatically characterized one.

Once such a "meta-analysis" of Turing's ways is given, one can try to see whether there are other mathematical concepts that have been analyzed in a similar way.[32] It seems to me that Dedekind's recasting of "geometric" continuity in "arithmetic" terms provides a convincing second example; the steps I will describe now are explicitly taken in [22]. The intended informal concept, "continuity of the geometric line", is *first* sharpened by

[32]Mendelson and Soare, for example, draw in their papers parallels between Turing's or Church's Thesis and other mathematical "theses". G. Kreisel has reflected on "informal rigor" generally and on its application to Church's Thesis in particular; a good presentation of Kreisel's views and a detailed list of his relevant papers can be found in [52].

the requirement that the line must not contain "gaps". The latter requirement is characterized, *second*, by the axiomatic condition that any "cut" of the line determines a geometric point. This "completeness" of the line is taken by Dedekind to be the "essence of continuity" and corresponds, as a normative demand, to Turing's central thesis. What corresponds to the *third* element in Turing's analysis, namely the rigorous proof? – Dedekind's argument, that the continuous geometric line and the system of rational cuts are isomorphic, does: the rationals can be associated with geometric points by fixing an origin on the line and a unit; the geometric cuts can then be transferred to the arithmetic realm. (To be sure, that requires the consideration of arbitrary partitions of the rationals satisfying the cut conditions and the proof that the system of rational cuts is indeed complete.) It is in this way that Dedekind's Thesis, or rather Dirichlet's demand that Dedekind tried to satisfy, is now supported: every statement of algebra and higher analysis can be viewed as a statement concerning natural numbers (and sets of such).[33]

Hilbert presented considerations concerning the continuum in his lectures from the Winter term 1919, entitled "Natur und mathematisches Erkennen"; he wanted to support the claim that

> the formation of concepts in mathematics is constantly guided by intuition and experience, so that on the whole mathematics is a non-arbitrary, unified structure.[34]

Having presented Dedekind's construction and his own investigation on non-Archimedean extensions of the rationals, he formulated the general point as follows:

> The different existing mathematical disciplines are consequently necessary parts in the construction of a systematic development of thought; this development begins with simple, natural questions and proceeds on a path that is essentially traced out by compelling internal reasons. There is no question of arbitrariness. Mathematics is not like a game that determines the tasks by arbitrarily invented rules, but rather a conceptual system of internal necessity that can only be thus and not otherwise.[35]

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[33]Let me add to the above analogy two further remarks: (i) both concepts are highly idealized—in Dedekind's case, he is clear about the fact that not all cuts are needed to have a model of Euclidean geometry, i.e., the constructibility of points is not a concern; for Turing, feasibility of computations is not a concern; (ii) both concepts are viewed by me as "abstract" mathematical concepts in the sense of my [59].

[34]l.c., p. 8; ... vielmehr zeigt sich, daß die Begriffsbildungen in der Mathematik beständig durch Anschauung und Erfahrung geleitet werden, so daß im großen und ganzen die Mathematik ein willkürfreies, geschlossenes Gebilde darstellt.

[35]l.c., p. 19: Es bilden also die verschiedenen vorliegenden mathematischen Disziplinen notwendige Glieder im Aufbau einer systematischen Gedankenentwicklung, welche von einfachen, naturgemäß sich bietenden Fragen anhebend, auf einem durch den Zwang innerer

Hilbert's remarks are fitting not only for the theory of the continuum, but also for the theory of computability.

Appendix A. Church's letter of June 8, 1937, to Pepis was enclosed with a letter to Bernays sent on June 14, 1937. Other material, also enclosed, were the "card" from Pepis to which Church's letter is a reply and the manuscript of Pepis's paper, *Ein Verfahren der mathematischen Logik*; Church asked Bernays to referee the paper. Church added, "Not because they are relevant to the question of acceptance of this particular paper, but because you may be interested in seeing the discussion of another point, I am sending you also a card received from Pepis and a copy of my reply. Please return Pepis's card when you write." In his letter of July 3, 1937, Bernays supported the publication of the paper which appeared in the 1938 edition of the Journal of Symbolic Logic; he also returned the card (which may very well be in the Church Nachlaß).

> Dear Mgr. [Monsignore] Pepis:
>
> This is to acknowledge receipt of your manuscript, *Ein Verfahren der mathematischen Logik*, offered for publication in the Journal of Symbolic Logic. In accordance with our usual procedure we are submitting this to a referee to determine the question of acceptance for publication, and I will write you further about the matter as soon as I have the referee's report.
>
> In reply to your postal [card] I will say that I am very much interested in your results on general recursiveness, and hope that I may soon be able to see them in detail. In regard to your project to construct an example of a numerical function which is effectively calculable but not general recursive I must confess myself extremely skeptical - although this attitude is of course subject to the reservation that I may be induced to change my opinion after seeing your work.
>
> I would say at the present time, however, that I have the impression that you do not fully appreciate the consequences which would follow from the construction of an effectively calculable non-recursive function.
>
> For instance, I think I may assume that we are agreed that if a numerical function f is effectively calculable then for every positive integer a there must exist a positive integer b such that a valid proof can be given of the proposition $f(a) = b$ (at least if we are not agreed on this then our ideas of effective calculability are so different as to

Gründe im wesentlichen vorgezeichneten Wege fortschreitet. Von Willkür ist hier keine Rede. Die Mathematik ist nicht wie ein Spiel, bei dem die Aufgaben durch willkürlich erdachte Regeln bestimmt werden, sondern ein begriffliches System von innerer Notwendigkeit, das nur so und nicht anders sein kann.

leave no common ground for discussion). But it is proved in my paper in the American Journal of Mathematics that if the system of Principia Mathematica is omega-consistent, and if the numerical function f is not general recursive, then, whatever permissible choice is made of a formal definition of f within the system of Principia, there must exist a positive integer a such that for no positive integer b is the proposition $f(a) = b$ provable within the system of Principia. Moreover this remains true if instead of the system of Principia we substitute any one of the extensions of Principia which have been proposed (e.g., allowing transfinite types), or any one of the forms of the Zermelo set theory, or indeed any system of symbolic logic whatsoever which to my knowledge has ever been proposed.

Therefore to discover a function which was effectively calculable but not general recursive would imply discovery of an utterly new principle of logic, not only never before formulated, but never before actually used in a mathematical proof - since all extant mathematics is formalizable within the system of Principia, or at least within one of its known extensions. Moreover this new principle of logic must be of so strange, and presumably complicated, a kind that its metamathematical expression as a rule of inference was not general recursive (for this reason, if such a proposal of a new principle of logic were ever actually made, I should be inclined to scrutinize the alleged effective applicability of the principle with considerable care).

Sincerely yours,
Alonzo Church

Appendix B. This is the part of Bernays's letter to Church of July 3, 1937, that deals with the latter's reply to Pepis.

Your correspondence with Mr. Pepis on his claimed discovery has much interested me. As to the consequence you draw from your result p. 357 Amer. Journ. Math., it seems to me that you have to use for it the principle of excluded middle. Without this there would remain the possibility that for the expression f it can *neither* be proved that to every μ standing for a positive integer m, there is a ν standing for a positive integer n such that the formula $\{f\}(\mu) = \nu$ is deducible within the logic, *nor* there can be denoted a positive integer m for which it can be proved that for no positive integer n the formula $\{f\}(\mu) = \nu$, where μ stands for m and ν for n, is deducible within the logic.

Appendix C. Church's review of Turing's paper in the Journal of Symbolic Logic is followed directly by his review of [54]:

The author proposes a definition of "finite 1-process" which is similar in formulation, and in fact equivalent, to computation by a Turing machine (see the preceding review). He does not, however, regard his formulation as certainly to be identified with effectiveness in the ordinary sense, but takes this identification as a "working hypothesis" in need of continual verification. To this the reviewer would object that effectiveness in the ordinary sense has not been given an exact definition, and hence the working hypothesis in question has not an exact meaning. To define effectiveness as computability by an arbitrary machine, subject to restrictions of finiteness, would seem to be an adequate representation of the ordinary notion, and if this is done the need for a working hypothesis disappears.

The present paper was written independently of Turing's, which was at the time in press but had not yet appeared.

Appendix D. On July 25, 1983 Church wrote a letter to John W. Dawson responding to the latter's inquiry, whether he (Church) had been "among those who thought that the Gödel incompleteness theorem might be found to depend on peculiarities of type theory". Church's letter is a rather touching (and informative) reflection on his work in the early thirties.

Dear Dr. Dawson:

In reply to your letter of June eighth, yes I was among those who thought that the Gödel incompleteness theorem might be found to depend on peculiarities of type theory (or, as I might later have added, of set theory) in a way that would show his results to have less universal significance than he was claiming for them. There was a historical reason for this, and that is that even before the Gödel results were published I was working on a project for a radically different formulation of logic which would (as I saw it at the time) escape some of the unfortunate restrictiveness of type theory. In a way I was seeking to do the very thing that Gödel proved impossible, and of course it's unfortunate that I was slow to recognize that the failure of Gödel's first proof to apply quite exactly to the sort of formulation of logic I had in mind was of no great significance.

The one thing of lasting importance that came out of my work in the thirties is the calculus of λ-conversion. And indeed this might be claimed as a system of logic to which the Gödel incompleteness theorem does not apply. To ask in what sense this claim is sound and in what sense not is not altogether pointless, as it may give some insight into the question where the boundary lies for applicability of the incompleteness theorem.

In my monograph on the calculus of λ-conversion (Annals of Mathematics Studies), in the section of the calculus of λ-δ-conversion (a minor variation of the λ-calculus) it is pointed out how, after identifying the positive integer 1 with the truth-value falsehood and the positive integer 2 with the truth-value truth, it is possible to introduce by definition, first the connectives of propositional calculus, and then an existential quantifier, but the latter only in the sense that: $(\exists x)M$ reduces to truth whenever there is some positive integer such that M reduces to truth after replacing x by the standard name of that positive integer, and in the contrary case $(\exists x)M$ has no normal form. The system is complete within its power of expression. But an attempt to introduce a universal quantifier, whether by definition or by added axioms, will give rise to some form of the Gödel incompletenes. - I'll not try to say more, as I am writing from recollection and haven't the monograph itself before me.

Sincerely,

Alonzo Church

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